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QUASI-HIERARCHICAL APPROACH TO DISCRETE MULTIOBJECTIVE STOCHASTIC DYNAMIC PROGRAMMING[3]

1. INTRODUCTION

Many decision problems are dynamic by their very nature. In such cases the decision is not made once, but many times. Partial choices are mutually related, since earlier decisions influence which decisions can be considered in the consecutive stages of the process.

The consequences of decisions become apparent in the near or remote future, which is uncertain by its very nature. Precise assessment of the results of the choices made is usually not possible. The information which is at the disposal of the decision maker is much more often incomplete and fragmentary. In such a situation he or she should, as far as possible, expand his/her knowledge of the problem under investigation. Although it is usually not possible to obtain data allowing to apply a deterministic model, these efforts can result in a partial knowledge thanks to which it is possible to estimate probability distributions describing values of the criteria obtained for the decision alternatives under consideration. In such cases we deal with what is called in the literature the problem of decision making under risk.

In such situations we can apply methods using discrete stochastic dynamic programming approach based on Bellman's optimality principle (Bellman, 1957). For these processes it is characteristic that at the beginning of each stage, the decision process is in a certain state. In each state, a set of feasible decisions is available. The process is discrete when all sets of states and decisions are finite. These processes are stochastic which means that the probability of achieving the final state for the given stage is known when at the beginning of this stage the process was in one of the admissible states and when a feasible decision has been made.

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We will consider additive multi-criteria processes. At each stage, we estimate the realisation of the process using stage criteria. The sum of the stage criteria gives the value of the multi-stage criterion. In the classical approach, the task consists in obtaining a strategy for which the expected value of the given criterion is optimal. Multi-criteria problems can be regarded as hierarchical problems. This means that the decision maker is able to formulate a hierarchy of criteria so that the most important criterion is assigned the number 1; the number 2 is reserved for the second-most important criterion, and so on. We assume that all criteria considered in the problem can be numbered in this way.

Usually we solve the hierarchical problem sequentially. First we find the set of solutions which are optimal with respect to the most important criterion. Out of this set, we select the subset of solutions which are optimal with respect to the criterion number 2. We continue this procedure until we determine the subset of solutions which are optimal with respect to the least important criterion.

The hierarchical approach has a certain essential shortcoming. It turns out that very often the subset of solutions, obtained when an important criterion in the hierarchy is considered, has only one element. As a result, the selection of the solution with respect to less important criteria is determined and these criteria do not play an essential role in the process of determining the final solution. It is why a quasi-hierarchy approach is proposed for solving hierarchical problems.

The quasi-hierarchical approach to hierarchical multi-objective stochastic programming seems quite new. Below we list some related theoretical and application papers.

Elmaghraby (1970) discusses some models most often encountered in Management Science applications: the shortest path problem between two specified nodes; the shortest distance matrix; as well as the special case of directed acyclic networks. One of related topics is finding the *k*-th shortest path.

The extension of the approach proposed above to discrete (deterministic) dynamic programming problem can be found in Trzaskalik (1990). The algorithm described there is applied to solve hierarchical deterministic dynamic programming problem.

Tempelmeier, Hilger (2015) consider the stochastic dynamic lot sizing problem with multiple items and limited capacity under two types of fill rate constraints. It is assumed that according to the static-uncertainty strategy, the production periods as well as the lot sizes are fixed in advance for the entire planning horizon and are executed regardless of the realisation of the demands.

Woerner et al. (2015) analyse Markov Decision Processes over compact state and action spaces. They investigate the special case of linear dynamics and piecewise-linear and convex immediate costs for the average cost criterion. This model is very general and covers many interesting examples, for instance in inventory management.

Shapiro (2012) analyse relations between the minimax, risk averse and nested formulations of multi-stage stochastic programming problems. In particular, it discusses conditions for time consistency of such formulations of stochastic problems.

Topaloglou et al. (2008) develop a multi-stage stochastic programming model for international portfolio management in a dynamic setting. They consider portfolio rebalancing decisions over multiple periods in accordance with the contingencies of the scenario tree. The solution jointly determines capital allocations to international markets, the selection of assets within each market, and appropriate currency hedging levels.

Hatzakis, Wallace (2006) describe a forward-looking approach for the solution of dynamic (time-changing) problems using evolutionary algorithms. The main idea of the proposed method is to combine a forecasting technique with an evolutionary algorithm. The location, in variable space, of the optimal solution (or of the Pareto optimal set in multi-objective problems) is estimated using a forecasting.

Dempster (2006) gives a comprehensive treatment of EVPI-based sequential importance sampling algorithms for multi-stage, dynamic stochastic programming problems. Both theory and computational algorithms are discussed.

Bakker et al. (2005) analyse the problem of robot planning (e.g. for navigation) with hierarchical maps. The authors present an algorithm for hierarchical path planning for stochastic tasks, based on Markov decision processes and dynamic programming.

Sethi et al. (2002) review the research devoted to proving that a hierarchy based on the frequencies of occurrence of different types of events in the systems results in decisions that are asymptotically optimal as the rates of some events become large compared to those of others. The paper also reviews the research on stochastic optimal control problems associated with manufacturing systems, their dynamic programming equations, existence of solutions of these equations, and verification theorems of optimality for the systems.

Sethi, Zhang (1994) present an asymptotic analysis of hierarchical manufacturing systems with stochastic demand and machines subject to breakdown and repair as the rate of change in machine states approaches infinity.

Daellenbach, De Kluyver (1980) present and illustrate a technique for finding MINSUM and MINMAX solutions to multi-criteria decision problems, called Multi Objective Dynamic Programming, capable of handling a wide range of linear, nonlinear, deterministic and stochastic multi-criteria decision problems. Multiple objectives are considered by defining an adjoin state space and solving an $(N + 1)$ terminal optimisation problem.

Two monographs Trzaskalik (1991, 1998) are also worth mentioning here. They present proposals for formulating and solving hierarchical problems of multiobjective dynamic programming approached deterministically.

In our paper we present a method based on a quasi-hierarchical approach. We assume that the decision maker is able to define a hierarchy of criteria and to determine the extent to which the optimal value of a higher-priority criterion can be made worse in order to improve the value of lower-priority criteria. To find the final solution of the problem, we start with determining the solutions for which the criteria take values no lower than the thresholds determined by the decision maker. Next, we use the criteria hierarchy to determine the optimal solution of the problem.

The main algorithm presented in this paper is based on the observation used previously in Trzaskalik (1991). Let us note that, except for the case of alternative solutions, when the optimal strategy is modified by changing the decision in any feasible process state, the expected value of the given criterion deteriorates. Therefore, it is necessary to consider all the strategies that differ from the optimal strategy in one of the feasible states and to select those which are within a determined tolerance interval. One should then analyse again the strategies found, changing the value in one of the feasible states. This process should be continued as long as it is possible to change the strategy for one of the feasible states, which provides a new strategy with the expected value of the realisation within the given tolerance interval.

The main idea of the approach discussed in the paper was previously presented on International Symposium of Management Engineering ISME 2015 Kitakyushu, Japan and International Conference of German, Austrian and Swiss Operations Research Societies (GOR, OGOR, SVOR/ASRO), University of Vienna, Austria, 2015 (Nowak, Trzaskalik, 2017). The final version of the paper, presented below includes full literature review, revised algorithms and detailed description of illustrative examples, not published before.

2. SINGLE-CRITERION STOCHASTIC DYNAMIC PROGRAMMING

We will use the following notation (Trzaskalik, 1991, 1998):

T – number of stages of the decision process under consideration,

y_t – state of the process at the beginning of stage t (t+1,…,T),

Y_t – finite set of process states at stage t,

x_t – feasible decision at stage t,

$X_t(y_t)$ – finite set of decisions feasible at stage t, when the process was in state $y_t \in Y_t$ at the beginning of this stage,

$F_t(y_{t+1} \mid y_t, x_t)$ – value of stage criterion at stage t for the transition from state y_t to state y_{t+1}, when the decision taken was $x_t \in X_t(y_t)$,

$P_t(y_{t+1} \mid y_t, x_t)$ – probability of the transition at stage t from state y_t to state y_{t+1}, when the decision taken was $x_t \in X_t(y_t)$.

$P(y_1)$ – probability of distribution in the set of initial stages $y_1 \in Y_1$.

The following holds:

$$\forall_{t\in\overline{1,T}} \forall_{y_t\in \mathbf{Y}_t} \forall_{x_t\in \mathbf{X}_t(y_y)} \sum_{y_{t+1}\in Y_{t+1}} P_t(y_{t+1} \mid y_t, x_t) = 1 \tag{1}$$

$\{x\}$ – strategy – a function assigning to each state $y_t \in Y_t$ exactly one decision $x_t \in X_t(y_t)$,

$\{X\}$ – the set of all strategies of the process under consideration,

$\{x_{\overline{t,T}}\}$ – shortened strategy, encompassing stages from t to T.

Let us assume that we have selected a certain strategy $\{\bar{x}\} \in \{X\}$. The expected value of this strategy is calculated as below.

Algorithm 1

1. For each state $y_T \in Y_T$ we calculate

$$G_T(y_T, \{\bar{x}_{\overline{T,T}}\}) = \sum_{y_{T+1} \in Y_{T+1}} F_T(y_{T+1} \mid y_T, \bar{x}_T) P_T(y_{T+1} \mid y_T, \bar{x}_T). \tag{2}$$

2. For each stage t, $t \in \overline{T-1,1}$ we calculate the expected value

$$G_t(y_t, \{\bar{x}_{\overline{t,T}}\}) = \sum_{y_{t+1} \in Y_{t+1}} (F_t(y_{t+1} \mid y_t, \bar{x}_t) + G_{t+1}(y_{t+1}, \{\bar{x}_{\overline{t+1,T}}\})) P_t(y_{t+1} \mid y_t, \bar{x}_t). \tag{3}$$

3. The expected value of the strategy $\{\bar{x}\} \in \{X\}$ is calculated from the formula:

$$G\{\bar{x}\} = \sum_{y_1 \in Y_1} G_1(y_1, \{\bar{x}\}) P_1(y_1). \tag{4}$$

Using Bellman's optimality principle (Bellman, 1957), we determine the optimal expected value for the process and optimal strategy.

Algorithm 2

1. For each state $y_T \in Y_T$ we calculate the optimal expected values

$$G_T^*(y_T) = \max_{x_T \in X_t(y_t)} \sum_{y_{T+1} \in Y_{T+1}} F_T(y_{T+1} \mid y_T, x_T) P_T(y_{T+1} \mid y_T, x_T) \tag{5}$$

   and find the decision $x_t^*(y_t)$, for which this maximum is attained. This decision forms a part of the optimal strategy being constructed.
2. For stage t, $t \in \overline{T-1,1}$ and each state $y_t \in Y_t$ we calculate the optimal expected values

$$G_t^*(y) = \max_{x_t \in X_t(y_t)} \sum_{y_{t+1} \in Y_{t+1}} (F_t(y_{t+1} \mid y_t, x_t) + G_{t+1}^*(y_{t+1})) P_t(y_{t+1} \mid y_t, x_t) \tag{6}$$

   and find the decision $x_t^*(y_t)$, for which this maximum is attained. This decision forms a part of the optimal strategy being constructed.
3. The optimal expected value of the process realisation is calculated from the formula:

$$G\{x^*\} = \sum_{y_1 \in Y_1} G_1^*(y_1, \{x^*\}) P_1(y_1). \tag{7}$$

3. DETERMINATION OF NEAR OPTIMAL STRATEGIES

The strategy $\{x^m\}$ is called near optimal if the expected value of its realisation differs from the expected value of the realisation of the optimal strategy $\{x^*\}$ by at most the given value z, that is

$$G\{x^*\} - G\{x^m\} \leq z \tag{8}$$

where $z > 0$.

We will use the following notation:

LS – the list of optimal and near optimal strategies,

LSB – the list of strategies to be investigated, that is of strategies which can be modified to determine further near optimal strategies,

LSC – the list of strategies considered in the algorithm,

M$\{x\}$ – the set of modified strategies which differ from the strategy $\{x\}$ by a decision in one state.

Algorithm 3

1. Set: LS:= $\varnothing$, LSB:= $\varnothing$, LSC = $\varnothing$.
2. Using **Algorithm 1** determine the set of strategies $\{X^*\}$, for which the given criterion attains the optimal value.
3. Add the strategies from the set $\{X^*\}$ to the sets LS and LSB:

   LS:= LS $\cup$ $\{X^*\}$,

   LSB := LBS $\cup$ $\{X^*\}$.
4. If LSB = $\varnothing$, go to step 11.
5. Select the next strategy $\{x\}$ from the set LSB; delete it from this set:

   LSB:= LSB \ $\{x\}$.
6. Determine all the modified strategies, which differ from the strategy $\{x\}$ by a decision taken in one state and add them to the set M$\{x\}$.
7. Check if the set M$\{x\}$ contains the strategies which are also in the sets LS, LSB and LSC. Delete the duplicate strategies from the set M$\{x\}$:

   M$\{x\}$ = M$\{x\}$ \ (M$\{x\}$ $\cap$ LS) \ (M$\{x\}$ $\cap$ LSB) \ M$\{x\}$ $\cap$ LSC.
8. Check if M$\{x\}$ $\neq \varnothing$. If not, go to step 4.
9. For the consecutive strategies $\{x^m\} \in$ M$\{x\}$:
   a) using formulas (2) i (3) calculate the expected value of the given criterion obtained by applying the strategy $\{x^m\}$,
   b) add the strategy $\{x^m\}$ to the set LSC:

      LSC:= LSC $\cup$ $\{x^m\}$,

c) if the expected value of the given criterion is no lower than Z, add the strategy $\{x^m\}$ to the sets LS and LSB:

$\text{LS} := \text{LS} \cup \{ x^m \},$

$\text{LSB} := \text{LSB} \cup \{ x^m \}.$

10. Go to step 4.
11. End of procedure.

This algorithm modifies the strategies from the set LSB by changing a decision in one state only.

For each new strategy we check if it generates a solution different from the ones determined previously. If so, we calculate the expected value of the given criterion and check if it satisfies the condition formulated by the decision maker. If this is not the case, such a strategy does not have to be further analysed, since its further modification cannot lead to an improvement of the criterion value. The procedure ends when the set LSB is empty.

4. APPLICATION OF THE QUASI-HIERARCHICAL APPROACH TO THE SOLUTION OF THE MULTIOBJECTIVE PROBLEM

Let us assume that the solution of the dynamic problem is evaluated with respect to K multi-stage criteria, each of which is the sum of T stage criteria. The evaluation of each strategy with respect to each criterion is based on the expected value. We assume that the decision maker ordered the criteria starting with the one he or she regards as the most important. We assume therefore that he is first of all interested in the optimisation of the criterion number 1, then of the criterion number 2, etc. The determination of the solution by means of the quasi-hierarchical approach is performed as follows:

Algorithm 4

1. Determine the optimal solutions of the problem with respect to each criterion.
2. Present the optimal values of each criterion to the decision maker.
3. Ask the decision maker to determine the aspiration thresholds Z_k, that is the values which should be attained by each criterion in the final solution.
4. For each criterion determine the set LS_k of strategies satisfying the requirements determined by the decision maker.
5. Set $J = K$.
6. Determine the set LS which is the intersection of the sets LS_k:

$$\text{LS} := \bigcap_{k \in \overline{1,J}} \text{LS}_k .$$

7. If $\text{LS} \neq \varnothing$, go to step 9.
8. Set $J := J - 1$. Go to step 6.

9. From among the solutions in the set LS select those for which the first criterion attains the highest value. If there are more than one such solutions, then in your selection take into account the values of the next criteria in the order determined by the hierarchy formulated by the decision maker.
10. If $LS := \bigcap_{k \in \overline{1,J}} LS_k = \varnothing$, check if the strategy obtained by this procedure satisfies the decision maker. If he is not satisfied, choose the final solution from the set LS not applying the proposition, formulated in 9. If he is still not satisfied, return to step 3.
11. End of procedure.

In the procedure we determine the set of alternatives which satisfy all the requirements determined by the decision maker. In many cases it may turn out that such solutions do not exist. In such cases we try to determine solutions satisfying the requirements formulated for those criteria, which the decision maker regards as the most important ones. Gradually, we therefore omit the requirements formulated for the least important criteria, until the set LS containing the solutions satisfying the requirements of the decision maker ceases to be empty. From among the solutions contained in this set we select the one for which the first criterion attains the highest value. If there are more than one such solutions, then in our selection we take into account the values of the next criteria according to the hierarchy defined by the decision maker.

5. ILLUSTRATIVE EXAMPLES

We will illustrate proposed method by means of illustrative examples. The description of the illustrative process under consideration can be found below.

We consider a three-stage decision process. The sets of states for the consecutive stages are as follows:

$$Y_1 = \{1,2\}, \quad Y_2 = \{3,4\}, \quad Y_3 = \{5,6\}.$$

We have the following set of final states of the process:

$$Y_4 = \{7,8\}.$$

The sets of feasible decisions are as follows:

$$X_1(1) = \{A, B\}, \qquad X_1(2) = \{C, D\},$$

$$X_2(3) = \{E, F\}, \qquad X_2(4) = \{G, H\},$$

$$X_3(5) = \{I, J\}, \qquad X_3(6) = \{K, L\}.$$

The graph of the process is given in figure 1. Rectangles denote states of the process in the consecutive stages, circles – random nodes.

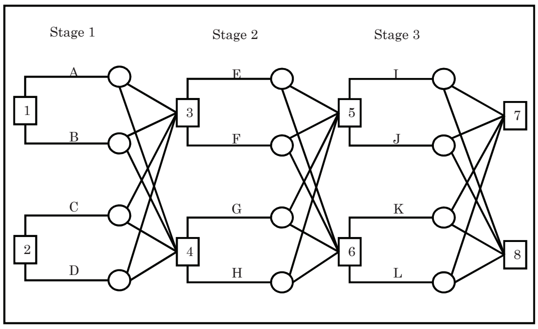

Figure 1. Graph of the process
Source: own elaboration.

The possible stage realisations of the process, probabilities of their occurrence, as well as the values of the stage criteria functions are shown in table 1.

Table 1.

Numeric values

Stage	$(y_{t+1}\|y_t, x_t)$	$P(\cdot)$	$F^1(\cdot)$	$F^2(\cdot)$	$F^3(\cdot)$	Stage	$(y_{t+1}\|y_t, x_t)$	$P(\cdot)$	$F^1(\cdot)$	$F^2(\cdot)$	$F^3(\cdot)$
1	(3|1,A)	0.4	6	15	22	2	(5|4,G)	0.6	5	15	20
1	(4|1,A)	0.6	8	17	14	2	(6|4,G)	0.4	6	18	13
1	(3|1,B)	0.7	6	15	22	2	(5|4,H)	0.8	5	15	20
1	(4|1,B)	0.3	8	17	14	2	(6|4,H)	0.2	6	18	13
1	(3|2,C)	0.5	6	15	22	3	(7|5,I)	0.8	5	30	12
1	(4|2,C)	0.5	8	17	14	3	(8|5,I)	0.2	1	12	15
1	(3|2,D)	0.8	6	15	22	3	(7|5,J)	0.3	5	30	12
1	(4|2,D)	0.2	8	17	14	3	(8|5,J)	0.7	1	12	15
2	(5|3,E)	0.5	5	15	20	3	(7|6,K)	0.2	5	30	12
2	(6|3,E)	0.5	6	18	13	3	(8|6,K)	0.8	1	12	15
2	(5|3,F)	0.3	5	15	20	3	(7|6,L)	0.9	5	30	12
2	(6|3,F)	0.7	6	18	13	3	(8|6,L)	0.1	1	12	15

Source: own data.

The probability distribution in the set of initial states is as follows:

$$P(1) = 0.4, \qquad P(2) = 0.6.$$

For clarity and due to small size of this illustrative problem, the existing strategies can be written down and numbered from 1 to 64. This numbering is presented in table 2.

Table 2.

List of strategies

No	Decision	No	Decision	No	Decision	No	Decision
1	(A,C,E,G,I,K)	17	(A,D,E,G,I,K)	33	(B,C,E,G,I,K)	49	(B,D,E,G,I,K)
2	(A,C,E,G,I,L)	18	(A,D,E,G,I,L)	34	(B,C,E,G,I,K)	50	(B,D,E,G,I,L)
3	(A,C,E,G,J,K)	19	(A,D,E,G,J,K)	35	(B,C,E,G,J,K)	51	(B,D,E,G,J,K)
4	(A,C,E,G,J,L)	20	(A,D,E,G,J,L)	36	(B,C,E,G,J,L)	52	(B,D,E,G,J,L)
5	(A,C,E,H,I,K)	21	(A,D,E,H,I,K)	37	(B,C,E,H,I,K)	53	(B,D,E,H,I,K)
6	(A,C,E,H,I,L)	22	(A,D,E,H,I,L)	38	(B,C,E,H,I,L)	54	(B,D,E,H,I,L)
7	(A,C,E,H,J,K)	23	(A,D,E,H,J,K)	39	(B,C,E,H,J,K)	55	(B,D,E,H,J,K)
8	(A,C,E,H,J,L)	24	(A,D,E,H,J,L)	40	(B,C,E,H,J,L)	56	(B,D,E,H,J,L)
9	(A,C,F,G,I,K)	25	(A,D,F,G,I,K)	41	(B,C,F,G,I,K)	57	(B,D,F,G,I,K)
10	(A,C,F,G,I,L)	26	(A,D,F,G,I,L)	42	(B,C,F,G,I,L)	58	(B,D,F,G,I,L)
11	(A,C,F,G,J,K)	27	(A,D,F,G,J,K)	43	(B,C,F,G,J,K)	59	(B,D,F,G,J,K)
12	(A,C,F,G,J,L)	28	(A,D,F,G,J,L)	44	(B,C,F,G,J,L)	60	(B,D,F,G,J,L)
13	(A,C,F,H,I,K)	29	(A,D,F,H,I,K)	45	(B,C,F,H,I,K)	61	(B,D,F,H,I,K)
14	(A,C,F,H,I,L)	30	(A,D,F,H,I,L)	46	(B,C,F,H,I,L)	62	(B,D,F,H,I,L)
15	(A,C,F,H,J,K)	31	(A,D,F,H,J,K)	47	(B,C,F,H,J,K)	63	(B,D,F,H,J,K)
16	(A,C,F,H,J,L)	32	(A,D,F,H,J,L)	48	(B,C,F,H,J,L)	64	(B,D,F,H,J,L)

Source: own elaboration.

Example 1

Applying Algorithm 1 find expected value for criterion F^1 and the strategy $\{x^{28}\}$. For t = 3 we have (formula (2)):

$$G_3(6,\{x^{28}\}) = F_3^1\,(7|6,L)\ P_3(7|6,L) + F_3^1(8|6,L)\ P_3(8|6,L) = 5\times0.9 + 1\times0.1 = 4.6,$$

$$G_3(5,\ \{x^{28}\}) = F_3^1(7|5,J)\times P_3(7|5,J) + F_3^1\,(8|5,J)\times P_3(8|5,J) = 5\times0.3 + 1\times0.7 = 2.2.$$

For t = 2 we have (formula (3)):

$$G_2(4,\{x^{28}\}) =$$
$$= [F_2^1(5|4,G) + G_3(5,\{x^{28}\}]\times P_2(5|4,G) + [F_2^1(6|4,G) + G_3(6,\{x^{28}\}]\times P_2(6|4,G) =$$
$$= (5+2.2)\times0.6 + (6+4.6)\times0.4 = 8.56,$$

$$G_2(3,\{x^{28}\}) =$$
$$= [F_2^1(5|3,F) + G_3(5,\{x^{28}\}]\times P_2\,(5|3,F) + [F_2^1(6|3,F) + G_3(6,\{x^{28}\}]\times P_2\,(6|3,F) =$$
$$= (5+2.2)\times0.3 + (6+4.6)\times0.7 = 9.58.$$

For $t = 1$ we have (formula (3)):

$G_1(2,\{x^{28}\}) =$
$= [F_1{}^1(3|2,\mathrm{D}) + G_2(3,\{x^{28}\}]\times P_1(3|2,\mathrm{D}) + [F_1{}^1(4|2,\mathrm{D}) + G_2(4,\{x^{28}\}]\times P_1(4|2,\mathrm{D}) =$
$= (6+9.58)\times 0.8 + (8+8.56)\times 0.2 = 15.776,$

$G_1(1,\{x^{28}\}) =$
$= [F_1{}^1(3|1,\mathrm{A}) + G_2(3,\{x^{28}\}]\times P_1(3|1,\mathrm{A}) + [F_1{}^1(4|1,\mathrm{A}) + G_2(4,\{x^{28}\}]\times P_1(4|1,\mathrm{A}) =$
$= (6+9.58)\times 0.4 + (8+8.56)\times 0.6 = 16.168.$

The expected value for the strategy $\{x^{28}\}$ is calculated from the formula (4):

$G(\{x^{28}\}) =$
$= G_1(1, \{x^{28}\})\times P(1) + G_1(2, \{x^{28}\})\times P(2) = 15.772\times 0.4 + 14.624\times 0.6 = 15.0832.$

Example 2

Applying Algorithm 2 find optimal expected values for criterion F^1 and optimal strategy for this criterion.

For $t = 3$ we have (formula (5)):

$G_3{}^*(6) = \max \{F_3{}^1(7|6,\mathrm{K})\times P_3(7|6,\mathrm{K}) + F_3{}^1(8|6,\mathrm{K})\times P_3(8|6,\mathrm{K}),$
$F_3{}^1(7|6,\mathrm{L})\times P_3(7|6,\mathrm{L}) + F_3{}^1(8|6,\mathrm{L})\times P_3(8|6,\mathrm{L})\} =$
$= \max \{(5\times 0.2 + 1\times 0.8), (5\times 0,9 + 1\times 0.1) = 4.6,$

$x_3{}^*(6) = \mathrm{L},$

$G_3{}^*(5) = \max \{F_3{}^1(7|5,\mathrm{I})\times P_3(7|5,\mathrm{I}) + F_3{}^1(8|5,\mathrm{I})\times P_3(8|5,\mathrm{I}),$
$F_3{}^1(7|5,\mathrm{J})\times P_3(7|5,\mathrm{J}) + F_3{}^1(8|5,\mathrm{J})\times P_3(8|5,\mathrm{J})\} =$
$= \max \{(5\times 0.8 + 1\times 0.2), 5\times 0,3 + 1\times 0.7) = 4.2,$

$x_3{}^*(5) = \mathrm{I}.$

For $t = 2$ we have (formula (6)):

$G_2{}^*(4) = \max \{[F_2{}^1(5|4,\mathrm{G}) + G_3{}^*(5)]\times P_2(5|4,\mathrm{G}) + [F_2{}^1(6|4,\mathrm{G}) + G_3{}^*(6)]\times P_2(|4,\mathrm{G}),$
$[F_2{}^1(5|4,\mathrm{H}) + G_3{}^*(5)]\times P_2(6|4,\mathrm{H}) + [F_2(6|4,\mathrm{H}) + G_3{}^*(6)]\times P_2(6|4,\mathrm{H})\} =$
$= \max \{[(5+4.2)\times 0.6 + (6+4.6)\times 0.4], [(5+4.2)\times 0.8 + (6+4.6)\times 0.2]\} = 9.76,$

$x_2{}^*(4) = \mathrm{G},$

$G_2{}^*(3) = \max \{[F_2{}^1(5|3,\mathrm{E}) + G_3{}^*(5)]\times P_2(5|3,\mathrm{E} + [F_2{}^1(6|3,\mathrm{E}) + G_3(6)]\times P_2(6|3,\mathrm{E}),$
$[F_2{}^1(5|3,\mathrm{E}) + G_3{}^*(5)]\times P_2(5|3,\mathrm{E}) + [F_2{}^1(6|3,\mathrm{E}) + G_3{}^*(6)]\times P_2(6|3,\mathrm{E})\} =$
$= \max \{[(5+4.2)\times 0.5 + (6+4.6)\times 0.5], [(5+4.2)\times 0.3 + (6+4.6)\times 0.7]\} = 9.76,$

$x_2{}^*(3) = \mathrm{F}.$

For t = 1 we have (formula (6)):

$$G_1^*(2) = \max \{[F_1^1(3|2,C) + G_2^*(3)]\times P_1(3|2,C) + [F_1^1(4|2,C) + G_2^*(4)]\times P_1(4|2,C), [F_1^1(3|2,D) + G_2^*(3)]\times P_1(3|2,D) + [F_1^1(4|2,D) + G_2^*(4)]\times P_1(4|2,D)\} =$$
$$= \max \{[(6+10.18)\times 0.5+(8+9.76)\times 0.5], [(6+10.18)\times 0.8+(8+9.76)\times 0.2]\} =$$
$$= 16.97,$$

$$x_1^*(2) = C,$$

$$G_1^*(1) = \max \{F_1^1(3|1,A) + G_2^*(3)]\times P_1(3|1,A) + [F_1^1(4|1,A) + G_2^*(4)]\times P_1(4|1,A), F_1^1(3|1,B) + G_2^*(3)]\times P_1(3|1,B) + [F_1^1(4|1,B) + G_2^*(4)]\times P_1(4|1,B)\} =$$
$$= \max \{[(6+10.18)\times 0.4+(8+9.76)\times 0.6], [(6+10.18)\times 0.7+(8+9.76)\times 0.3]\} =$$
$$= 17.128,$$

$$x_1^*(1) = A.$$

The optimal expected value is calculated from the formula (7):

$$G\{x^*\} = G_1^*(1)\times P(1) + G_1^*(2)\times P(2) = 17.128\times 0.4 + 16.97\times 0.6 = 17.0332.$$

Example 3

We have to determine all the strategies for which the expected value of the criterion number 1 differs from the optimal value by at most 2%.

The determination of near optimal strategies, described in **Algorithm 3** proceeds as follows:

1. We set

   LS:= ∅, LSB:= ∅, LSC = ∅.

2. Using Algorithm 1 we find the set optimal strategy:

   $\{X^*\} = \{\{x^{10}\}\}$

   for which the given criterion attains the optimal value equal to 17.0332. We have (see table 2):

   $\{x^{10}\}$ = (A,C,F,G.I,L).

3. We add the strategy found to the sets LS and LSB. We have:

   LS = LS ∪ $\{X^*\} = \{\{x^{10}\}\}$,

   LSB = LSB ∪ $\{X^*\} = \{\{x^{10}\}\}$.

4. Since LSB ≠ ∅, we go to step 5.
5. We select the strategy $\{x^{10}\}$ from the set LSB and delete it from this set:

   LSB:= LSB \ $\{x^{10}\}$ = ∅.

6. We determine all the modified strategies which differ from the strategy $\{x^{10}\}$ by a decision taken in one state:
   $\{x^9\}$ = { A,C,F,G,I,K}, $\{x^{12}\}$ = { A,C,F,G,J,L }, $\{x^{14}\}$ = { A,C,F,H,I,L },
   $\{x^2\}$ = { A,C,E,G,I,L }, $\{x^{26}\}$ = { A,D,F,G,I,L }, $\{x^{42}\}$ = { B,C,F,G,I,L },
   and add them to the set M$\{x^{10}\}$:
   M$\{x^{10}\} = \{\{x^9\}, \{x^{12}\}, \{x^{14}\}, \{x^2\}, \{x^{26}\}, \{x^{42}\}\}$.
7. We check if the set M$\{x^{10}\}$ contains strategies which are also contained in the sets LS, LSB and LSC. We obtain:
   M$\{x^{10}\} \cap$ LS = $\varnothing$,
   M$\{x^{10}\} \cap$ LSB = $\varnothing$,
   M$\{x^{10}\} \cap$ LSC = $\varnothing$.
8. We have M$\{x^{10}\} \neq \varnothing$.
9. We consider further strategies $\{x^m\} \in$ M$\{x^{10}\}$.

Strategy $\{x^9\}$

a) we calculate the expected value
   $G\{x^9\} = 15.5268$,
b) we add the strategy $\{x^9\}$ to the set LSC:
   LSC:= LSC $\cup \{x^9\} = \{\{x^9\}\}$,
c) since
   $G\{x^9\} = 15.5268 < 16.6925$,
   we do not add the strategy $\{x^9\}$ to the sets LS and LSB.

Strategy $\{x^{12}\}$

a) we calculate the expected value
   $G\{x^{12}\} = 16.192$,
b) we add the strategy $\{x^{12}\}$ to the set LSC:
   LSC:= LSC $\cup \{x^{12}\} = \{\{x^9\}, \{x^{12}\}\}$,
c) since
   $G\{x^{12}\} = 16.192 < 16.6925$,
   we do not add the strategy $\{x^{12}\}$ to the set LS or to the set LSB.

Strategy $\{x^{14}\}$

a) we calculate the expected value
   $G\{x^{14}\} = 16.882$,

b) we add the strategy $\{x^{14}\}$ to the set LSC:

LSC:= LSC $\cup$ $\{x^{14}\}$ = $\{\{x^9\}, \{x^{12}\}, \{x^{14}\}\}$,

c) since

$G\{x^{14}\}$ = 16.882 > 16.6925,

we add the strategy $\{x^{14}\}$ to the sets LS and LSB:

LS:= LS $\cup$ $\{x^{14}\}$ = $\{\{x^{10}\}, \{x^{14}\}\}$,

LSB:= LSB $\cup$ $\{x^{14}\}$ = $\{\{x^{14}\}\}$.

Strategy $\{x^2\}$

a) we calculate the expected value

$G\{x^2\}$ = 16.9044,

b) we add the strategy $\{x^2\}$ to the set LSC:

LSC:= LSC $\cup$ $\{x^2\}$ = $\{\{x^9\}, \{x^{12}\}, \{x^{14}\}, \{x^2\}\}$,

c) since

$G\{x^{63}\}$= 16.9044 > 16.6925,

we add the strategy $\{x^2\}$ to the set LS and to the set LSB:

LS:= LS $\cup$ $\{x^2\}$ = $\{\{x^{10}\}, \{x^{14}\}, \{x^2\}\}$,

LSB:= LSB $\cup$ $\{x^{14}\}$ = $\{\{x^{14}\}, \{x^2\}\}$.

Strategy $\{x^{26}\}$

a) we calculate the expected value

$G\{x^{26}\}$ = 16.7488,

b) we add the strategy $\{x^2\}$ to the set LSC:

LSC:= LSC $\cup$ $\{x^{26}\}$ = $\{\{x^9\}, \{x^{12}\}, \{x^{14}\}, \{x^2\}, \{x^{26}\}\}$,

c) since

$G\{x^{26}\}$ = 16.7488 > 16.6925,

we add the strategy $\{x^2\}$ to the set LS and to the set LSB:

LS:= LS $\cup$ $\{x^2\}$ = $\{\{x^{10}\}, \{x^{14}\}, \{x^2\}, \{x^{26}\}\}$,

LSB:= LSB $\cup$ $\{x^{14}\}$ = $\{\{x^{14}\}, \{x^2\}, \{x^{26}\}\}$.

Strategy $\{x^{42}\}$

a) we calculate the expected value

$G\{x^2\}$ = 16.8436,

b) we add the strategy $\{x^{42}\}$ to the set LSC:

LSC:= LSC $\cup$ $\{x^2\}$ = $\{\{x^9\}, \{x^{12}\}, \{x^{14}\}, \{x^2\}, \{x^{42}\}\}$,

c) since

$G\{x^{63}\} = 16.8436 > 16.6952$,

we add the strategy $\{x^2\}$ to the set LS and to the set LSB:

LS:= LS $\cup$ $\{x^2\}$ = $\{\{x^{10}\}, \{x^{14}\}, \{x^2\}, \{x^{42}\}\}$,

LSB:= LSB $\cup$ $\{x^{14}\}$ = $\{\{x^{14}\}, \{x^2\}, \{x^{42}\}\}$.

10. We go to step 4.
4. Since LSB $\neq \varnothing$, we go to step 5.

Continuing this procedure, in the consecutive steps we determine the next strategies which are near optimal and satisfy the condition:

$G\{x\} > 16.6952$.

The set of optimal and near optimal strategies, for which the expected value of the given criterion differs from the optimal value by no more than 2%, that is for which $G\{x\} \geq 16.6952$, contains the following strategies:

LS = $\{\{x^{10}\}, \{x^{14}\}, \{x^2\}, \{x^{42}\}, \{x^6\}, \{x^{26}\}, \{x^{46}\}\}$.

Example 4

Now we regard the considered process as a three-criteria hierarchical process, in which the most important is the first criterion, the second-most important is the second criterion, and the least important is the third criterion.

The determination of the final strategy using the quasi-hierarchical procedure described in **Algorithm 4** is performed as follows:

1. We determine the optimal solution of the problem with respect to each criterion. The optimal strategy with respect to the first criterion is $\{x^{10}\}$. The expected value for this strategy with respect to the first criterion is 17.0332.
   The optimal strategy with respect to the second criterion is $\{x^{10}\}$. The expected value for this strategy with respect to the second criterion is 60.0624.
   The optimal strategy with respect to the third criterion is $\{x^{55}\}$. The expected value for this strategy with respect to the third criterion is 51.3124.
2. We present the optimal values of each criterion to the decision maker.
3. Based on the information obtained, the decision maker decided, that the expected values of all the criteria can differ from the optimal value at most 2%. It means, that the aspiration levels are as follows:
   - Z_1 = 16.6925 for criterion 1,
   - Z_2 = 58.8612 for criterion 2,
   - Z_3 = 50.2862 for criterion 3.

4. For each criterion we determine the set of strategies satisfying the requirements determined by the decision maker. We obtain:

   $LS_1 = \{\{x^{10}\}, \{x^{14}\}, \{x^{2}\}, \{x^{42}\}, \{x^{6}\}, \{x^{26}\}, \{x^{46}\}\}$,

   $LS_2 = \{\{x^{10}\},\{x^{42}\},\{x^{26}\},\{x^{58}\},\{x^{62}\},\{x^{2}\},\{x^{30}\},$
   $\{x^{46}\},\{x^{14}\},\{x^{34}\},\{x^{18}\},\{x^{50}\}, x^{6}\},\{x^{38}\},\{x^{22}\},\{x^{54}\}\}$,

   $LS_3 = \{\{x^{55}\},\{x^{51}\},\{x^{23}\},\{x^{53}\},\{x^{56}\}\}$.

5. We set J =3.
6. We determine

   $LS_1 \cap LS_2 \cap LS_3 = \varnothing$.

7. Since LS = ∅, we go to step 8.
8. We set $J := J - 1 = 2$. We go to step 6.
6. We determine

   $LS = LS_1 \cap LS_2 = \{\{x^{10}\},\{x^{2}\},\{x^{14}\},\{x^{42}\}, \{\{x^{26}\},\{x^{46}\}\}$.

7. Since LS ≠ ∅, we go to step 9.
9. From among the solutions from the set LS we select the strategy $\{x^{10}\}$, which is optimal for both the first and the second criteria.
10. Since for $J = 3$ we have $LS := \bigcap_{k \in \overline{1,J}} LS_k = \varnothing$ we check if the strategy obtained in our procedure satisfies the decision maker. The expected value of the strategy $\{x^{10}\}$ for criterion 3 is 46.3526. It is easy to check that this is the worst strategy with respect to the expected value for criterion 3, hence the selection of this strategy as the final strategy is very unsatisfactory for the decision maker. We return to step 3.

3 The decision maker decided to slightly lower the aspiration level with respect to the first criterion. The new aspiration level with respect to criterion 1 is
$Z_1 = 16.68$.

4. We determine the sets:

   $LS_1 = \{\{x^{10}\},\{x^{14}\},\{x^{2}\},\{x^{42}\},\{x^{6}\},\{x^{26}\},\{x^{46}\}, \{x^{34}\}\}$,

   $LS_2 = \{\{x^{10}\},\{x^{42}\},\{x^{26}\},\{x^{58}\},\{x^{62}\},\{x^{2}\},\{x^{30}\},\{x^{46}\},\{x^{14}\},\{x^{34}\},\{x^{18}\},\{x^{50}\},$
   $\{x^{6}\},\{x^{38}\},\{x^{22}\},\{x^{54}\}\}$,

   $LS_3 = \{\{x^{55}\},\{x^{51}\},\{x^{23}\},\{x^{53}\},\{x^{56}\}\}$.

5. We set $J = 3$.
6. We have

   $LS_1 \cap LS_2 \cap LS_3 = \varnothing$.

7. Since LS = ∅, we go to step 8.
8. We set $J := J - 1 = 2$. We go to step 6.

6. We determine

   $LS = LS_1 \cap LS_2 .= \{\{x^{10}\},\{x^{14}\},\{x^2\},\{x^{42}\}, \{x^{26}\},\{x^{46}\}, \{x^{34}\}\}$.
7. Since $LS \neq \varnothing$, we go to step 9.
9. From among the solutions belonging to the set LS the best strategy is again $\{x^{10}\}$, disqualified previously by the decision maker. That is why we consider the next strategies from the set LS_3.
10. We present the strategy $\{x^{34}\}$, for which $G^3\{x^{34}\}= 47.8966$, to the decision maker. This value, although still far from the aspiration level accepted by the decision maker for the third criterion, has been accepted by him.
11. End of procedure.

6. SUMMARY

The quasi-hierarchical approach is a frequently used method of solving multi-criteria problems. It requires that the decision maker order the criteria of the evaluation of the decision process. In our paper we have presented a way of applying this approach to solving the multi-criteria problem. It has been assumed that the decision maker is able to order the criteria from the most important one to the least important one, and that based on the information about optimal solutions with respect to each criterion he or she can formulate the conditions to be satisfied by the strategies to be taken into account in the determination of the final solution of the problem.

In our paper we assumed that the evaluation of the quality of the individual solutions with respect to each criterion was based on the expected value. This is not, however, the only possible way of analysing the problem. For the evaluations of the solutions one can use also measures based on the probability of the occurrence of a given event, as well as the conditional expected value. In future papers we intend to propose a quasi-hierarchical method taking into account criteria of this type.

The approach proposed in the paper can be applied to solve a variety of problems. In future research, we are going to show how the quasi-hierarchical dynamic approach can be used to solve the project portfolio selection problem and production capacity planning problem.

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PODEJŚCIE QUASI-HIERARCHICZNE W DYSKRETNYM WIELOKRYTERIALNYM STOCHASTYCZNYM PROGRAMOWANIU DYNAMICZNYM

Streszczenie

W pracy rozważany jest wieloetapowy wielokryterialny proces podejmowania decyzji w warunkach ryzyka. W celu jego rozwiązania wykorzystano dyskretne stochastyczne programowanie dynamiczne oparte na zasadzie optymalności Bellmana. Zakłada się, że decydent jest w stanie zdefiniować quasi-hierarchię rozważanych kryteriów, co oznacza, że jest on w stanie określić w jakim zakresie optymalna wartość oczekiwana dla kryteriów o wyższym priorytecie może być pogorszona w celu poprawy wartości oczekiwanej kryterium o priorytecie niższym. Proces uzyskania rozwiązania końcowego może być realizowany interaktywnie. Obserwując kolejno proponowane rozwiązania, decydent może modyfikować poziomy aspiracji dla rozważanych kryteriów, otrzymując ostatecznie rozwiązanie satysfakcjonujące. Metoda została zilustrowana przykładem opartym na danych umownych.

Słowa kluczowe: programowanie dynamiczne, podejmowanie decyzji w warunkach ryzyka, podejście interaktywne, metoda quasi-hierarchiczna

QUASI-HIERARCHICAL APPROACH TO DISCRETE MULTIOBJECTIVE STOCHASTIC DYNAMIC PROGRAMMING

Abstract

In this paper we consider a multi-stage, multi-criteria discrete decision process under risk. We use a discrete, stochastic dynamic programming approach based on Bellman's principle of optimality. We assume that the decision maker determines a quasi-hierarchy of the criteria considered; in other words, he or she is able to determine to what extent the optimal expected value of a higher-priority criterion can be made worse to improve the expected value of a lower-priority criterion. The process of obtaining the final solution can be interactive. Based on the observations of the consecutive solutions, the decision maker can modify the aspiration levels with respect to the criteria under consideration, finally achieving a solution which satisfies him/her best. The method is illustrated on an example based on fictitious data.

Keywords: dynamic programming, decision making under risk, interactive approach, quasi-hierarchical method