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Multi-asset portfolio with trapezoidal fuzzy present values[3]

1. INTRODUCTION

By a financial asset we understand an authorization to receive a future financial revenue, payable to a certain maturity. The value of this revenue is interpreted as anticipated future value (FV) of the asset. According to the uncertainty theory (von Mises, 1962; Kaplan, Barish, 1967), any unknown future state is uncertain. This uncertainty stems from of our lack of knowledge about the future. Yet, in the researched case, we can point out this particular time in the future, in which the considered income value will be already known to the observer. After Kolmogorov (1933, 1956), von Mises (1957), Lambalgen (1996), Sadowski (1976, 1980), Czerwiński (1960, 1969), Caplan (2001) we will accept this as a sufficient condition for modelling the uncertainty with probability. All this leads to a conclusion that FV is a random variable.

The main focus of following research is present value (PV), defined as a present equivalent of a payment available in a given time in the future. PV of future cash flows is widely accepted to be an approximate value, with fuzzy numbers being one of the main tools of its modelling. Ward (1985) defined fuzzy PV as a discounted fuzzy forecast of a future cash flow's value. Fuzzy numbers were introduced into financial arithmetic by Buckley (1987). As a result, Ward's definition was then further generalized by Greenhut et. al (1995), Sheen (2005) and Huang (2007), who expands Ward's definition to the case of a future cash flow given as a fuzzy variable. More general definition of fuzzy PV was proposed by

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Tsao (2005), who assumes that future cash flow can be treated as a fuzzy probabilistic set. All those authors depict PV as a discounted, imprecisely estimated future cash flow. A different approach was given by Piasecki (2011, 2014), where fuzzy PV was estimated by a current market value of the financial asset.

Piasecki (2011) showed that if the PV of an asset is a fuzzy real number, then its return rate is a fuzzy probabilistic set (Hirota, 1981). Works of Buckley (1987), Gutierrez (1989), Kuchta (2000) and Lesage (2001) have previously proved the sensibility of using triangular or trapezoidal fuzzy numbers as a fuzzy financial arithmetic tool.

In Siwek (2015) two cases of a simple two-asset portfolio with fuzzy triangular and trapezoidal present value were researched. After Markowitz (1952), both articles assume normal distribution of simple return rates, with fuzzy expected return rate being the main tool of an asset assessment. Nonetheless, the results of performed research were highly complicated in forms of energy and entropy measures for the portfolio expected return rate, which made it difficult to continue researching the topic.

In Piasecki, Siwek (2017) an alternative approach is suggested to solve the problem researched in Siwek (2015). The fuzzy discount factor is used for appraising the financial instrument with triangular fuzzy PV. Regretfully, entropy measure of an arbitrary triangular fuzzy number is constant, which makes it difficult to analyze the impact of the diversification on imprecision of a portfolio assessment. On the other hand, trapezoidal fuzzy numbers do not have this disadvantage. Thus, the main purpose of presented article is to generalize these results to the case where PVs of portfolio assets are given by trapezoidal fuzzy numbers. In addition, in our considerations, the two-asset portfolio will be replaced with a more general multiple asset portfolio. This way we can extend the possibility of managing the risk burdening a multi-asset portfolio, constructed with use of an imprecise information stemming from present value of component assets.

2. ELEMENTS OF FUZZY NUMBER THEORY

By $\mathcal{F}(\mathbb{R})$ we denote a family of all fuzzy subsets of a real line $\mathbb{R}$. Dubois, Prade (1979) define a fuzzy number as a fuzzy subset $K \in \mathcal{F}(\mathbb{R})$, represented by membership function $\mu_K \in [0;1]^{\mathbb{R}}$ which[4] satisfies following conditions

$$\exists_{x\in\mathbb{R}}\, \mu_K(x) = 1, \tag{1}$$

[4] Symbol $[0;1]^{\mathbb{R}}$ denotes the family of all function from the real line $\mathbb{R}$ into the interval $[0,1]$.

$$\forall_{(x,y,z)\in\mathbb{R}^3}\colon\ x \le y \le z \Longrightarrow \mu_K(y) \ge \min\{\mu_K(x), \mu_K(z)\}. \tag{2}$$

The set of all fuzzy numbers will be denoted as $\mathbb{F}$. Arithmetic operations on fuzzy numbers were defined by Dubois, Prade (1978). By $\circledcirc$ we denote the ordinary arithmetic operation $\circ$ generalized to the case of fuzzy numbers. According to the Zadeh's Extension Principle (Zadeh, 1965), a sum of fuzzy numbers $K, L \in \mathbb{F}$ represented by their corresponding membership functions $\mu_K, \mu_L \in [0;1]^{\mathbb{R}}$ is a fuzzy subset

$$M = K \oplus L \tag{3}$$

described by its membership function $\mu_M \in [0;1]^{\mathbb{R}}$

$$\mu_M(z) = \sup\{\mu_K(x) \wedge \mu_L(z-x)\colon x \in \mathbb{R}\}. \tag{4}$$

The sum of the sequence $\{K_i\}_{i=1}^{n} \subset \mathbb{F}$ we denote by

$$\bigoplus_{i=1}^{n} K_i = K_1 \oplus K_2 \oplus \dots \oplus K_n. \tag{5}$$

Analogously, the multiplication of a real number $y \in \mathbb{R}^+$ and a fuzzy number $K \in \mathbb{F}$ represented by membership function $\mu_K \in [0;1]^{\mathbb{R}}$ is a fuzzy subset

$$N = y \odot K \tag{6}$$

described by its membership function $\mu_N \in [0;1]^{\mathbb{R}}$

$$\mu_N(z) = \mu_K\left(\frac{z}{y}\right). \tag{7}$$

Moreover, if $y = 0$, then the multiplication (6) is equal to zero. The class of fuzzy real numbers is closed under the operations (3) and (6). Our further research will be limited to the case of fuzzy numbers with bounded support.

Fuzzy numbers are widely used for modeling assessments or estimations of a parameter given imprecisely. After Klir (1993) we understand imprecision as a superposition of ambiguity and indistinctness of information. Ambiguity can be interpreted as a lack of a clear recommendation between one alternative among various others. Indistinctness is understood as a lack of explicit distinction between recommended and not recommended alternatives. An increase in information imprecision makes it less useful and therefore it is logical to consider the problem of imprecision assessment.

We measure the ambiguity of a fuzzy number by applying the Khalili's measure (1979) to the energy measure $d\colon \mathcal{F}(\mathbb{R}) \to \mathbb{R}_0^+$ defined by de Luca, Termini (1979). For an arbitrary fuzzy number $K \in \mathbb{F}$, with membership function $\mu_K \in [0;1]^{\mathbb{R}}$ we have

$$d(K) = \int_{-\infty}^{+\infty} \mu_K(x)dx. \tag{8}$$

The indistinctness of an arbitrary fuzzy number can be measured by its entropy $e\colon \mathcal{F}(\mathbb{R}) \to \mathbb{R}_0^+$, also defined by de Luca, Termini (1972) and in form given by Kosko (1986). For an arbitrary fuzzy number $K \in \mathbb{F}$ we have

$$e(K) = \frac{d(K \cap K^C)}{d\big((K \cup K^C) \cap \mathbb{S}(K)\big)}, \tag{9}$$

where $\mathbb{S}(K) = \{x \in \mathbb{R}: \mu_K(x) > 0\}$.

The main focus of this study is a trapezoidal fuzzy number. The fuzzy number $Tr(r,s,t,u)$, defined for a non-decreasing sequence $\{r,s,t,u\} \subset \mathbb{R}$ with a membership function $\mu(\cdot\,|r,s,t,u) \in [0,1]^{\mathbb{R}}$ given by the formula

$$\mu(x|r,s,t,u) = \begin{cases} 0 & \text{for} \quad x < r, \\ \dfrac{x-r}{s-r} & \text{for} \quad r \le x < s, \\ 1 & \text{for} \quad s \le x \le t, \\ \dfrac{x-u}{t-u} & \text{for} \quad t < x \le u, \\ 0 & \text{for} \quad x > u, \end{cases} \tag{10}$$

is a trapezoidal fuzzy number.

For any arbitrary pair of trapezoidal fuzzy numbers, $Tr(r_1,s_1,t_1,u_1)$ and $Tr(r_2,s_2,t_2,u_2)$ and $a,b \in \mathbb{R}_0^+$ we have:

$$\begin{aligned} &Tr(ar_1+br_2, as_1+bs_2, at_1+bt_2, au_1+bu_2) = \\ &= \big(a \odot Tr(r_1,s_1,t_1,u_1)\big) \oplus \big(b \odot Tr(r_2,s_2,t_2,u_2)\big), \end{aligned} \tag{11}$$

$$d\big(Tr(r_1,s_1,t_1,u_1)\big) = \frac{1}{2}(u_1+t_1-r_1-s_1), \tag{12}$$

$$e\big(Tr(r_1,s_1,t_1,u_1)\big) = \frac{s_1-r_1+u_1-t_1}{-s_1-3r_1+3u_1+t_1}. \tag{13}$$

3. RETURN RATE FROM A FINANCIAL ASSET

All calculations in this article are performed for a fixed time $t > 0$. We use simple return rates r_t defined as

$$r_t = \frac{V_t - V_0}{V_0}, \tag{14}$$

where:

— V_t is a *FV* described by random variable $\tilde{V}_t \colon \Omega \to \mathbb{R}$;
— V_0 is a *PV* assessed precisely or approximately.

Variable *FV* is described by a relation

$$\tilde{V}_t(\omega) = \check{C} \cdot \left(1 + \tilde{r}_t(\omega)\right), \tag{15}$$

where the simple return rate $\tilde{r}_t \colon \Omega \to \mathbb{R}$ is determined for *PV* equal to the market price $\check{C}$. After Markowitz (1952) we assume that $\tilde{r}$ rate has a normal probability distribution $N(\bar{r}, \sigma)$.

Moreover, in the researched case we assume that the *PV* is estimated by a trapezoidal fuzzy number $Tr\left(\check{C}_{min}, \check{C}_*, \check{C}^*, \check{C}_{max}\right)$, determined by membership function $\mu \in [0,1]^{\mathbb{R}}$ described by (10). This condition was initially introduced by Kuchta (2000) and applied in Siwek (2015). Parameters of the trapezoidal fuzzy number $Tr\left(\check{C}_{min}, \check{C}_*, \check{C}^*, \check{C}_{max}\right)$ are interpreted as follows:

— $\check{C}$ is the market price,
— $\check{C}_{min} \in \left]0, \check{C}\right]$ is the maximal lower bound of *PV*,
— $\check{C}_{max} \left[\check{C}, +\infty\right[$ is the minimal upper bound of *PV*,
— $\check{C}_* \in \left[\check{C}_{min}, \check{C}\right]$ is the minimal upper assessment of prices visibly lower than the market price $\check{C}$,
— $\check{C}^* \in \left[\check{C}, \check{C}_{max}\right]$ is the maximal lower assessment of prices visibly higher than the market price $\check{C}$.

Methods of determining parameters $\check{C}_{min}, \check{C}_{max}$ are given in Piasecki, Siwek (2015). These parameters are non-negative.

According to the Zadeh's Extension Principle, a simple return rate for *PV* given as a trapezoidal fuzzy number is a fuzzy probabilistic set with membership function $\tilde{\rho} \in [0; 1]^{\mathbb{R}\times\Omega}$

$$\tilde{\rho}(r, \omega) = \sup\left\{\mu\left(x \middle| \check{C}_{min}; \check{C}_*; \check{C}^*; \check{C}_{max}\right) \colon x = \frac{\tilde{V}_t(\omega)}{1 + r}, x \in \mathbb{R}\right\} = \tag{16}$$

$$= \mu\left(\frac{\tilde{V}_t(\omega)}{1+r}\middle|\check{C}_{min};\check{C}_*;\check{C}^*;\check{C}_{max}\right) = \mu\left(\check{C}\,\frac{1+\tilde{r}_t(\omega)}{1+r}\middle|\check{C}_{min};\check{C}_*;\check{C}^*;\check{C}_{max}\right).$$

According to (10), formula (16) can be transformed into

$$\rho(r,\omega) = \begin{cases} \dfrac{\check{C}\,\dfrac{1+\tilde{r}_t(\omega)}{1+r} - \check{C}_{min}}{\check{C}_* - \check{C}_{min}}, & \text{for } \check{C}_{min} \le \check{C}\,\dfrac{1+\tilde{r}_t(\omega)}{1+r} < \check{C}_*, \\ 1, & \text{for } \check{C}_* \le \check{C}\,\dfrac{1+\tilde{r}_t(\omega)}{1+r} < \check{C}^*, \\ \dfrac{\check{C}\,\dfrac{1+\tilde{r}_t(\omega)}{1+r} - \check{C}_{max}}{\check{C}^* - \check{C}_{max}}, & \text{for } \check{C}^* < \check{C}\,\dfrac{1+\tilde{r}_t(\omega)}{1+r} \le \check{C}_{max}, \\ 0, & \text{for } \check{C}\,\dfrac{1+\tilde{r}_t(\omega)}{1+r} > \check{C}_{max}, \check{C}\,\dfrac{1+\tilde{r}_t(\omega)}{1+r} < \check{C}_{min}. \end{cases} \tag{17}$$

Thus, the expected return rate $R \in \mathbb{F}$ is a fuzzy number with membership function $\rho \in [0,1]^{\mathbb{R}}$:

$$\rho(r) = \begin{cases} \dfrac{\check{C}\,\dfrac{1+\bar{r}}{1+r} - \check{C}_{min}}{\check{C}_* - \check{C}_{min}}, & \text{for } \check{C}_{min} \le \check{C}\,\dfrac{1+\bar{r}}{1+r} < \check{C}_*, \\ 1, & \text{for } \check{C}_* \le \check{C}\,\dfrac{1+\bar{r}}{1+r} < \check{C}^*, \\ \dfrac{\check{C}\,\dfrac{1+\bar{r}}{1+r} - \check{C}_{max}}{\check{C}^* - \check{C}_{max}}, & \text{for } \check{C}^* < \check{C}\,\dfrac{1+\bar{r}}{1+r} \le \check{C}_{max}, \\ 0, & \text{for } \check{C}\,\dfrac{1+\bar{r}}{1+r} > \check{C}_{max}, \check{C}\,\dfrac{1+\bar{r}}{1+r} < \check{C}_{min}. \end{cases} \tag{18}$$

The expected discount factor $\bar{v}$ calculated using the return rate $\bar{r}$ is given by identity

$$\bar{v} = \frac{1}{1+\bar{r}} \tag{19}$$

which is its definition. Therefore, the function $\delta \in [0;1]^{\mathbb{R}}$ described by

$$\delta(v) = \delta\left(\frac{1}{1+r}\right) = \rho(r) \tag{20}$$

is a membership function of the expected discount factor $D \in \mathbb{F}$ calculated using the expected return rate $R \in \mathbb{F}$. Combining both (18) and (20) we get

$$\delta(v) = \begin{cases} \frac{\check{C}v - \bar{v}\check{C}_{min}}{\bar{v}\check{C}_* - \bar{v}\check{C}_{min}} & \text{for} \quad \frac{\bar{v}}{\check{C}}\check{C}_{min} \le v \le \frac{\bar{v}}{\check{C}}\check{C}_*, \\ 1 & \text{for} \quad \frac{\bar{v}}{\check{C}}\check{C}_* < v < \frac{\bar{v}}{\check{C}}\check{C}^*, \\ \frac{\check{C}v - \bar{v}\check{C}_{max}}{\bar{v}\,\check{C}^* - \bar{v}\check{C}_{max}} & \text{for} \quad \frac{\bar{v}}{\check{C}}\check{C}^* < v \le \frac{\bar{v}}{\check{C}}\check{C}_{max}, \\ 0 & \text{for} \quad \frac{\bar{v}}{\check{C}}\check{C}_{max} < v, \qquad v < \frac{\bar{v}}{\check{C}}\check{C}_{min}. \end{cases} \tag{21}$$

One can see that the expected fuzzy discount factor stated above is also a trapezoidal fuzzy number $Tr\left(\check{C}_{min}\frac{\bar{v}}{\check{C}}, \check{C}_*\frac{\bar{v}}{\check{C}}, \check{C}^*\frac{\bar{v}}{\check{C}}, \check{C}_{max}\frac{\bar{v}}{\check{C}}\right)$.

An increase in ambiguity of expected discount factor $D \in \mathbb{F}$ suggests a higher number of alternative recommendations to choose from. This may result in making a decision, which will be *ex post* associated with a profit lower than maximal, that is with a loss of chance. This kind of risk is called an ambiguity risk. The ambiguity risk of D is measured by energy measure $d(D)$.

An increase in the indistinctness of D, on the other hand, suggests that the differences between recommended and not recommended decision alternatives are harder to differentiate. This leads to an increase in the indistinctness risk, that is in the risk of choosing a not recommended option. The indistinctness risk of an expected discount factor D is measured by entropy measure $e(D)$. Imprecision risk consists of both ambiguity and indistinctness risk, combined.

From (15) we have that the return rate is a function of the future value of an asset, which is uncertain, since we don't know the future state of the world. Because of this, the investor is not sure whether they will gain or lose from the decision made. With the increase in uncertainty, the risk of making a wrong decision is higher. Here, uncertainty risk of a return rate will be measured by its variance σ^2.

As compared to energy and entropy measures of a return rate form a portfolio with component assets given imprecisely by a triangular or trapezoidal fuzzy number (Siwek, 2015), the simplicity of those measures calculated for discount factors encourages their use in analyzing portfolios burdened with imprecision. The criterion of minimizing the discounting factor may then substitute the criterion of return rate maximization, with same theoretical conclusions.

4. MULTI-ASSET PORTFOLIO

By a financial portfolio we will understand an arbitrary, finite set of financial assets. Each of this assets is characterized by its assessed PV and anticipated return rate.

Let us consider the case of a multi-asset portfolio π, consisting of financial assets Y_i $(i = 1,2, \ldots, n)$. The PV of assets Y_i is estimated by fuzzy trapezoidal number $Tr\left(\check{C}_{min}^{(i)}, \check{C}_{*}^{(i)}, \check{C}^{*(i)}, \check{C}_{max}^{(i)}\right)$ where parameters are given as follows:

— $\check{C}^{(i)}$ is the market price,
— $\check{C}_{min}^{(i)} \in \left]0, \check{C}^{(i)}\right]$ is the maximal lower bound of PV,
— $\check{C}_{max}^{(i)} \in \left[\check{C}^{(i)}, +\infty\right[$ is the minimal upper bound of PV,
— $\check{C}_{*}^{(i)} \in \left[\check{C}_{min}^{(i)}, \check{C}^{(i)}\right]$ is the minimal upper assessment of prices visibly lower than the market price $\check{C}^{(i)}$,
— $\check{C}^{*(i)} \in \left[\check{C}^{(i)}, \check{C}_{max}^{(i)}\right]$ is the maximal lower assessment of prices visibly higher than the market price $\check{C}^{(i)}$.

We assume that for each security Y_i we know the simple return rate $\tilde{r}_t^i : \Omega \to \mathbb{R}$ appointed by (14) for the PV equal to the market price $\check{C}^{(i)}$. After Markowitz (1952) we assume that the n-dimensional variable $(\tilde{r}_t^1, \tilde{r}_t^2, \ldots, \tilde{r}_t^n)^T$ has a cumulative normal distribution $N(\bar{\boldsymbol{r}}, \boldsymbol{\Sigma})$ where $\bar{\boldsymbol{r}} = (\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_n)^T$. We appoint an expected discount factor of security Y_i:

$$D^{(i)} = Tr\left(\check{C}_{min}^{(i)} \frac{\bar{v}_i}{\check{C}^{(i)}}, \check{C}_{*}^{(i)}, \check{C}^{*(i)} \frac{\bar{v}_i}{\check{C}^{(i)}}, \check{C}_{max}^{(i)} \frac{\bar{v}_i}{\check{C}^{(i)}}\right), \tag{22}$$

where $\bar{v}_i$ is an expected discount factor appointed using the expected return rate $\bar{r}_i$. According to (12), the energy measure of $D^{(i)}$ is given by

$$d\left(D^{(i)}\right) = \frac{\overline{v_i}}{2\check{C}^i}\left(\check{C}^{*(i)} + \check{C}_{max}^{(i)} - \check{C}_{min}^{(i)} - \check{C}_{*}^{(i)}\right), \tag{23}$$

and from (13), the entropy measure of a discounting factor can be calculated as

$$e\left(\, D^{(i)}\right) = \frac{\check{C}^{*(i)} - \check{C}_{min}^{(i)} + \check{C}_{max}^{(i)} - \check{C}_{*}^{(i)}}{-\check{C}_{*}^{(i)} - 3\check{C}_{min}^{(i)} + 3\check{C}_{max}^{(i)} + \check{C}^{*(i)}}. \tag{24}$$

We have that the market value $\check{C}^{(\pi)}$ of a portfolio π is equal to

$$\check{C}^{(\pi)} = \textstyle\sum_{i=1}^{n} \check{C}^{(i)}. \tag{25}$$

Share p_i of an instrument Y_i in the portfolio π is given by

$$p_i = \frac{\check{C}^{(i)}}{\check{C}^{(\pi)}}. \tag{26}$$

We denote $\overline{\boldsymbol{p}} = (p_1, p_2, \ldots, p_n)^T$. Then expected portfolio return rate $\bar{r}$ equals

$$\bar{r} = \bar{\boldsymbol{p}}^T \bar{\boldsymbol{r}}. \tag{27}$$

As for the present value of the portfolio, according to (11) it is also a trapezoidal fuzzy number

$$PV^{(\pi)} = Tr\left(\sum_{i=1}^{n} \check{C}_{min}^{(i)}, \sum_{i=1}^{n} \check{C}_{*}^{(i)}, \sum_{i=1}^{n} \check{C}^{*(1)}, \sum_{i=1}^{n} \check{C}_{max}^{(i)}\right) = \\ = Tr\left(\check{C}_{min}^{(\pi)}, \check{C}_{*}^{(\pi)}, \check{C}^{*(\pi)}, \check{C}_{max}^{(\pi)}\right). \tag{28}$$

By (20), one can calculate the fuzzy expected discount factor of the portfolio π:

$$D^{(\pi)} = Tr\left(\check{C}_{min}^{(\pi)} \frac{\bar{v}}{\check{C}^{(\pi)}}, \check{C}_{*}^{(\pi)} \frac{\bar{v}}{\check{C}^{(\pi)}}, \check{C}^{*(\pi)} \frac{\bar{v}}{\check{C}^{(\pi)}}, \check{C}_{max}^{(\pi)} \frac{\bar{v}}{\check{C}^{(\pi)}}\right), \tag{29}$$

where $\bar{v}$ is a discounting factor calculated for expected return rate $\bar{r}$. We have

$$\frac{1}{\bar{v}} = \sum_{i=1}^{n} \frac{p_i}{\bar{v}_i}, \tag{30}$$

from which we obtain:

$$\bar{v} = \left(\sum_{i=1}^{n} \frac{p_i}{\bar{v}_i}\right)^{-1} = \left(\sum_{i=1}^{n} \frac{p_i}{\bar{v}_i}\right)^{-1} \sum_{i=1}^{n} p_i = \left(\sum_{i=1}^{n} \frac{p_i}{\bar{v}_i}\right)^{-1} \sum_{i=1}^{n} \frac{p_i}{\bar{v}_i} \bar{v}_i, \tag{31}$$

$$\frac{\bar{v}}{\check{C}^{(\pi)}} \check{C}_{min}^{(\pi)} = \frac{\bar{v}}{\check{C}^{(\pi)}} \sum_{i=1}^{n} \check{C}_{min}^{(i)} = \bar{v} \sum_{i=1}^{n} p_i \frac{\check{C}_{min}^{(i)}}{\check{C}^{(i)}} = \\ = \left(\sum_{i=1}^{n} \frac{p_i}{\bar{v}_i}\right)^{-1} \sum_{i=1}^{n} \frac{p_i}{\bar{v}_i} \left(\bar{v}_i \frac{\check{C}_{min}^{(i)}}{\check{C}^{(i)}}\right), \tag{32}$$

$$\frac{\bar{v}}{\check{C}^{(\pi)}} \check{C}_{*}^{(\pi)} = \frac{\bar{v}}{\check{C}^{(\pi)}} \sum_{i=1}^{n} \check{C}_{*}^{(i)} = \bar{v} \sum_{i=1}^{n} p_i \frac{\check{C}_{*}^{(i)}}{\check{C}^{(i)}} = \\ = \left(\sum_{i=1}^{n} \frac{p_1}{\bar{v}_1}\right)^{-1} \sum_{i=1}^{n} \frac{p_i}{\bar{v}_i} \left(\bar{v}_i \frac{\check{C}_{*}^{(i)}}{\check{C}^{(i)}}\right), \tag{33}$$

$$\frac{\bar{v}}{\check{C}^{(\pi)}} \check{C}^{*(\pi)} = \frac{\bar{v}}{\check{C}^{(\pi)}} \sum_{i=1}^{n} \check{C}^{*(i)} = \bar{v} \sum_{i=1}^{n} p_i \frac{\check{C}^{*(i)}}{\check{C}^{(i)}} = \\ = \left(\sum_{i=1}^{n} \frac{p_i}{\bar{v}_i}\right)^{-1} \sum_{i=1}^{n} \frac{p_i}{\bar{v}_i} \left(\bar{v}_1 \frac{\check{C}^{*(i)}}{\check{C}^{(i)}}\right), \tag{34}$$

$$\frac{\bar{v}}{\check{C}^{(\pi)}}\check{C}_{max}^{(\pi)} = \frac{\bar{v}}{\check{C}^{(\pi)}}\sum_{i=1}^{n}\check{C}_{max}^{(i)} = \bar{v}\sum_{i=1}^{n}p_i\frac{\check{C}_{max}^{(i)}}{\check{C}^{(i)}} = \\ = \left(\sum_{i=1}^{n}\frac{p_i}{\bar{v}_i}\right)^{-1}\sum_{i=1}^{n}\frac{p_i}{\bar{v}_i}\left(\bar{v}_i\frac{\check{C}_{max}^{(i)}}{\check{C}^{(i)}}\right). \tag{35}$$

From the formulas given above, we can rewrite the fuzzy discount factor as

$$D^{(\pi)} = \left(\sum_{i=1}^{n}\frac{p_i}{\bar{v}_i}\right)^{-1} \odot \left(\oplus_{i=1}^{n}\frac{p_i}{\bar{v}_i}\odot D^{(i)}\right). \tag{36}$$

From (12) and (36), we obtain that the energy measure of an expected discounting factor $D \in \mathbb{F}$ is a linear combination of energy measures calculated for each of component assets

$$d\left(D^{(\pi)}\right) = \left(\sum_{i=1}^{n}\frac{p_i}{\bar{v}_i}\right)^{-1}\sum_{i=1}^{n}\frac{p_i}{\bar{v}_i}d\left(D^{(i)}\right). \tag{37}$$

The relation above suggests that the energy of a fuzzy expected discount factor of a portfolio π is, in fact, a linear combination of weighted energies of those factors calculated for its components. The weights calculated for the assets Y_i increase with their shares in the portfolio and, respectively, decrease with the value of their discount factor $\bar{v}_i$. This fact leads to a conclusion, that when trying to minimize the ambiguity risk of a portfolio, one should focus on minimizing the ambiguity of component assets, which are characterized by the highest expected return rates. On the other hand, the shares of an asset in the whole portfolio are, according to the theory, appointed *post factum,* by gathering available information on said assets. Condition (37) shows that, in the researched case, the portfolio diversification only ”averages” the risk of ambiguity.

According to (13), the entropy measure of expected discount factor is equal

$$e\left(D^{(\pi)}\right) = \frac{\check{C}_{*}^{(\pi)} - \check{C}_{min}^{(\pi)} + \check{C}_{max}^{(\pi)} - \check{C}^{*(\pi)}}{-\check{C}_{*}^{(\pi)} - 3\check{C}_{min}^{(\pi)} + 3\check{C}_{max}^{(\pi)} + \check{C}^{*(\pi)}}. \tag{38}$$

The variance of a portfolio return rate is calculated by

$$\sigma^2 = \overline{\boldsymbol{p}}^T\boldsymbol{\Sigma}\overline{\boldsymbol{p}}. \tag{39}$$

By constructing a portfolio which minimizes the variance Markowitz proved that portfolio diversification can ”minimize” the uncertainty risk.

5. CASE STUDY

The portfolio π consists of financial assets Y_1 and Y_2. Anticipated vector $(\tilde{r}_t^1, \tilde{r}_t^2)^T$ of their simple return rates has two-dimensional normal distribution

$$N\left((0.25{,}0.5)^T, \begin{bmatrix} 0.5 & -0.1 \\ -0.1 & 0.4 \end{bmatrix}\right).$$

For the asset Y_1 with market price $\check{C}^{(1)} = 24$, its PV is estimated by a trapezoidal fuzzy number $Tr(18, 23, 25, 37)$. Then according to (18), the fuzzy expected return rate from Y_1 is a fuzzy number $R_1 \in \mathbb{F}$ given by membership function $\rho_1 \in [0,1]^{\mathbb{R}}$

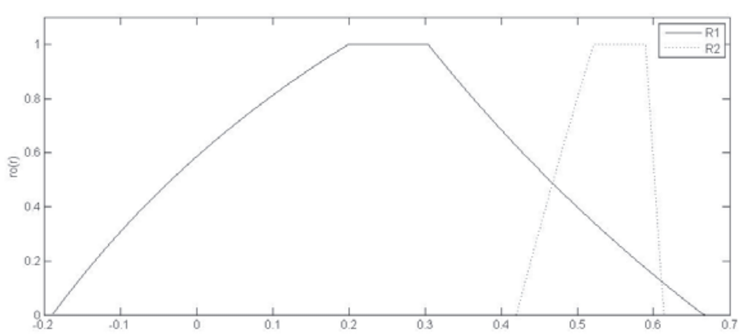
$$\rho_1(r) = \begin{cases} \dfrac{6}{1+r} - 3.6, & \text{for} \quad 0.67 \geq r > 0.30, \\ 1, & \text{for} \quad 0.30 \geq r > 0.20, \\ \dfrac{-2{,}5}{1+r} + 3.08 & \text{for} \quad 0.20 > r \geq -0.19, \\ 0 & \text{for} \quad r \notin [-0.19, \quad 0.67]. \end{cases}$$

For the asset Y_2 with market price $\check{C}^{(2)} = 69$, its PV is estimated by fuzzy trapezoidal number $Tr(66, 67, 70, 75)$. Then according to (18), the fuzzy expected return rate of Y_2 is a fuzzy number $R_2 \in \mathbb{F}$ with membership function $\rho_2 \in [0,1]^{\mathbb{R}}$

$$\rho_2(r) = \begin{cases} \dfrac{103.5}{1+r} - 66, & \text{for} \quad 0.57 \geq r > 0.54, \\ 1, & \text{for} \quad 0.54 \geq r > 0.48, \\ \dfrac{-20.7}{1+r} + 15 & \text{for} \quad 0.48 > r \geq 0.38, \\ 0 & \text{for} \quad r \notin [0.38, 0.57]. \end{cases}$$

Membership functions for return rates of both assets are presented in figure 1. It is easy to see that the expected return rate is not a trapezoidal fuzzy number.

Figure 1. Membership functions for expected return rates R_1 and R_2



Source: own study.

Using the expected return rate R_1 we may now appoint by means of (21) a fuzzy expected discount factor $D^{(1)} \in \mathbb{F}$. We have

$$D^{(1)} = Tr\left(18\frac{0.8}{24}, 23\frac{0.8}{24}, 25\frac{0.8}{24}, 37\frac{0.8}{24}\right) = Tr(0.6, 0.77, 0.83, 1.23).$$

According to (29), its energy measure equals

$$d\left(D^{(1)}\right) = \frac{0,8}{2 \cdot 24}(37 - 18 + 25 - 23) = 0.35.$$

and from (30), the entropy measure has the value of

$$e\left(D^{(1)}\right) = \frac{23 - 18 + 37 - 25}{-23 - 3 \cdot 18 + 3 \cdot 37 + 25} = 0.29.$$

The expected discount factor $D^{(2)} \in \mathbb{F}$ of the second asset calculated using R_2 equals

$$D^{(2)} = Tr\left(66\frac{0,67}{69}, 67\frac{0,67}{69}, 70\frac{0,67}{69}, 75\frac{0,67}{69}\right) = Tr(0.64,0.6,0.68,0.73).$$

Also, its energy measure equals

$$d\left(D^{(2)}\right) = \frac{0,67}{2 \cdot 69}(75 - 66 + 70 - 67) = 0.06.$$

and entropy measure has the value of

$$e\left(D^{(2)}\right) = \frac{67 - 66 + 75 - 70}{-67 - 3 \cdot 66 + 3 \cdot 75 + 70} = 0.2.$$

The market price of portfolio π is equal to

$$\check{C}^{(\pi)} = 24 + 69 = 93.$$

Corresponding to (26), shares p_1 and p_2 of Y_1 and Y_2 in the portfolio π are equal

$$p_1 = \frac{24}{93}, \quad p_2 = \frac{69}{93}.$$

We can appoint the fuzzy expected discount factor $D^{(\pi)} \in \mathbb{F}$ of the portfolio π. By (36), it is a fuzzy number of the form

$$D^{(\pi)} = \left(\left(\left(\frac{\frac{24}{93}}{0.8} + \frac{\frac{69}{93}}{0.67}\right)^{-1} \cdot \frac{\frac{24}{93}}{0.8}\right) \odot D^{(1)}\right) \oplus$$

$$\oplus \left(\left(\left(\frac{\frac{24}{93}}{0.8} + \frac{\frac{69}{93}}{0.67}\right)^{-1} \cdot \frac{\frac{69}{93}}{0.67}\right) \odot D^{(2)}\right) =$$

$$= \left(0.2256 \odot D^{(1)}\right) \oplus \left(0.7744 \odot D^{(2)}\right) = Tr(0.63, 0.68, 0.71, 0.84).$$

Its energy measure calculated by (37) equals

$$d\left(D^{(\pi)}\right) = 0.2256 \cdot 0.35 + 0.7744 \cdot 0.06 = 0.13.$$

Entropy measure can be calculated by (38)

$$e\left(D^{(\pi)}\right) = e\left(Tr(0.63, 0.68, 0.71, 0.84)\right) = \frac{0.68 - 0.63 + 0.84 - 0.71}{-0.68 - 3 \cdot 0.63 + 3 \cdot 0.84 + 0.71} = 0{,}27.$$

Let us note that we have

$$\left(\frac{p_1}{\bar{v}_1} + \frac{p_2}{\bar{v}_2}\right)^{-1} \left(\frac{p_1}{\bar{v}_1} e\left(D^{(1)}\right) + \frac{p_2}{\bar{v}_2} e\left(D^{(2)}\right)\right) =$$

$$= 0.2256 \cdot 0.29 + 0.7744 \cdot 0.2 = 0.2203 \neq 0{,}27 = e\left(D^{(\pi)}\right).$$

It implies that the portfolio entropy measure $e\left(D^{(\pi)}\right)$ cannot be calculated similarly to the portfolio energy measure $e\left(D^{(\pi)}\right)$ by the linear combination (37).

We obtain following relations between the energy measure and entropy measure appointed for fuzzy expected discount factors of portfolio and its components:

$$d(D_1) > d\left(D^{(\pi)}\right) > d(D_2),$$

$$e(D_1) > e\left(D^{(\pi)}\right) > e(D_2).$$

These inequalities show that the portfolio diversification can average the imprecision risk. Moreover, using (39) we calculate the variance of a return rate from portfolio:

$$\sigma^2 = 0.2175.$$

By increasing the number of assets in the portfolio, we can lower the variance (which approaches its limit with number of assets going to infinity). This means that creating a multi asset portfolio π results in minimizing the uncertainty risk.

Let us consider now any portfolio π consisting of financial assets Y_1 and Y_2. The contribution of the instrument Y_i in the portfolio π is equal to p_i. Then, according to (36), the expected discount factor $D^{(\pi)} \in \mathbb{F}$ of the portfolio π can be calculated in the following way

$$D^{(\pi)} = \left(\frac{p_1}{0.8} + \frac{p_2}{0.67}\right)^{-1} \odot$$
$$\odot \left(\left(\frac{p_1}{0.8} \odot Tr(0.6, 0.77, 0.83, 1.23) \right) \oplus \left(\frac{p_2}{0.67} \odot Tr(0.64; 0.65; 0.68; 0.73) \right) \right) =$$
$$= \frac{0.67p_1 \odot Tr(0.6, 0.77, 0.83, 1.23) \oplus 0.8p_2 \odot Tr(0.64; 0.65; 0.68; 0.73)}{0.667p_1 + 0.8p_2} =$$
$$= \frac{p_1 \odot Tr(0.402, 0.5159, 0.5561, 0.8241) \oplus p_2 \odot Tr(0.512, 0.52, 0.544, 0.584)}{0.67p_1 + 0.8p_2}.$$

We see that the expected fuzzy discount factor of portfolio can be expressed as a combination of securities contributions and their expected fuzzy discount factors. In an analogous way the ambiguity risk may be evaluated because of the energy measure for this factor by (37) is given as follows:

$$d\left(D^{(\pi)}\right) = \left(\frac{p_1}{0.8} + \frac{p_2}{0.67}\right)^{-1} \left(\frac{p_1}{0.8} 0.35 + \frac{p_2}{0.67} 0.06\right) = \frac{0.67p_1 0.35 + 0.8p_2 0.06}{0.67p_1 + 0.8p_2} =$$
$$= \frac{0.2345p_1 + 0.048p_2}{0.67p_1 + 0.8p_2}.$$

Above we have shown that entropy measure $e\left(D^{(\pi)}\right)$ cannot be expressed in analogous way. The last two equations can be applied to the mathematical programing task dedicated to portfolio optimization.

6. SUMMARY

The main purpose of this article was to analyse the possibility of managing the risk burdening a two-asset portfolio, built with use of an imprecise information stemming from present value of component assets. The imprecise present values were modelled with by trapezoidal fuzzy numbers. For this assumptions we have reached the following conclusions:

— The portfolio diversification can lower uncertainty risk,
— The portfolio diversification averages ambiguity risk,
— The portfolio diversification can to average indistinctness risk.

The results obtained suggest, on one hand, that the portfolio diversification does not help in lowering the imprecision risk, but on the other hand, it also does not increase it. Thus, research suggests that there exist portfolios, which imprecision risk will not be minimized with portfolio diversification, and thus it is vital to create a new risk minimization problem, including all of the risk types.

The results obtained above encourage for their broader analysis. Further research can focus on generalizing the representation of the present value to an arbitrary fuzzy number. Helpful here can be fundamental results obtained in Goetschel, Voxman (1986) and Stefaninia et al. (2006).

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PORTFEL WIELOSKŁADNIKOWY Z TRAPEZOIDALNYMI ROZMYTYMI WARTOŚCIAMI BIEŻĄCYMI

Streszczenie

Głównym celem niniejszego artykułu jest przedstawienie charakterystycznych cech portfela wieloskładnikowego w przypadku, kiedy bieżące wartości składników portfela są trapezoidalnymi liczbami rozmytymi. W ramach analizy portfelowej jest wyznaczany rozmyty oczekiwany czynnik dyskonta i oceny ryzyka nieprecyzyjności. Dzięki temu pojawia się możliwość opisania wpływu dywersyfikacji portfela na ryzyko nieprecyzyjności. Przedstawione teoretyczne rozważania i uzyskane wnioski są poparte przykładem liczbowym.

Słowa kluczowe: portfel wieloskładnikowy, wartość bieżąca, trapezoidalna liczba rozmyta, czynnik dyskontowy

MULTI-ASSET PORTFOLIO WITH TRAPEZOIDAL FUZZY PRESENT VALUES

Abstract

The main purpose of the following paper is to present characteristics of a multi-asset portfolio in case of present values of composing financial instruments being modelled by a trapezoidal fuzzy number. Throughout the analysis a fuzzy expected discount factor and imprecision risk assessments are calculated. Thanks to that, there arises a possibility to describe the influence of portfolio diversification on imprecision risk. Presented theoretical inference and obtained conclusions are supported by numerical example.

Keywords: multi-asset portfolio, present value, trapezoidal fuzzy number, discount factor