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Strong and safe Nash equilibrium in some repeated 3-player games^{4,5}

1. INTRODUCTION

The Prisoner's Dilemma with its generalizations are very important as an example of conflicts and social dilemmas. As we can find in Dawes (1980), social dilemmas are real life problems which have two properties: "1. each individual receives a higher payoff for a socially defection choice than for a socially cooperative choice, no matter what the other individuals in society do; 2. all individuals are better off if all cooperate than if all defect." An example of such situation in real life is a problem of soldiers who fight in a battle. They are personally better off taking no chances, yet if no one fight against the enemy, then the result will be worst for all of soldiers. Such dilemmas can be found among resource depletion, pollution and overpopulation.

Social dilemmas are games in which there is a conflict between individual rationality and optimality of the equilibrium payoff. Since it is observable that people cooperate with each other in the real situations, game theorists have faced the obstacle, how to construct simple tools to encourage players in such games to cooperate with each other. The model need to approximate the real situation and strategies should be likely to use.

A natural approach is to consider the infinitely repeated game. Usually, all players observe the whole history of action profiles used in previous stages of the repeated game. Such situation is called the game with complete information.

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The strategies are functions from the set of the histories into the set of actions. Payoffs in the repeated game are either the discounted sum of stage payoffs or the limit of average payoffs. The aim of this approach is to obtain the Nash equilibrium in the repeated game with the pair of payoffs which is close to the cooperation payoffs in the stage game. Since the fifties of the last century there appeared various folk theorems which was not explicitly published and, in many cases, the original author is unknown.

The classic Prisoner's Dilemma is a 2-player game, in which each player has two actions, usually denoted as C (cooperation) and D (defection). The game has a unique Nash equilibrium – a pair of actions such that the action of each of the players optimize this player's payoff given the action of the opponent. The Nash equilibrium is the action profile (D, D) which is the pair of strictly dominant actions i.e. playing D is better than C whatever the other player does. What is more, both players benefit changing (D, D) into (C, C) . So, the mechanism of individual rationality fails in the Prisoner's Dilemma and it leads to a loss of both players. It means that the Nash equilibrium is not Pareto-optimal in this case.

One of solutions for lack of cooperation of the Nash equilibrium in the stage game is an idea of good strategies introduced in Smale (1980) for the repeated Prisoner's Dilemma. Every pair of good strategies is a Nash equilibrium in the repeated game with Pareto-optimal payoffs corresponding to the payoff of (C, C) in the stage game. The second advantage of the good strategies equilibrium is the warranted minimal payoff for the non-deviating player. The minimal payoff is equal to the Nash payoff in the stage game. Good strategies have yet another advantage that has not been pointed in Smale (1980). Choosing a good strategy appropriately, the player controls the second player's payoff. For every $\varepsilon > 0$ there exists the ε -good strategy of the first player such that for an arbitrary second player's strategy, the first player's payoff will be at most ε smaller than the second player's payoff. The Prisoner's Dilemma is symmetric, so the second player also can choose the ε -good strategy which provides him no worse payoffs than the first player's one minus ε .

In fact, good strategies have properties that was postulated in Axelrod (1984). In 80's he studied the evolution of cooperation. It refers to how cooperation can emerge and persist as elucidated by application of game theory. He organized a tournament in which game theory experts submitted their strategies and each strategy was paired with each other for 200 iterations of Prisoner's Dilemma. Accumulated payoffs through the tournament was treated as a score. The winner was the strategy submitted by Arnold Rappaport – Tit for Tat. The additional advantage of this tournament was detecting what properties strategies should satisfy to encourage players to cooperate. They should be: nice, forgiving, retaliatory and are founded on simple rules. Good strategies have these properties and, what is more, player cooperates until the other player's average payoff is greater than his average payoff plus ε . By choosing ε , the player determines the level of his tolerance for the defection.

In this paper we shall consider the generalization of the idea of good strategies onto the Prisoner's Dilemma type repeated game for three players. We consider the repeated game with a partial monitoring. We assume that, after each stage, all players can only observe an aggregated history – the arithmetic mean of the payoffs from previous stages. The stage game is a symmetric 3-player game where each player has an action set consisting of two actions: I and NI ⁶. We assume that the action profile (NI, NI, NI) is the only Nash equilibrium, and the sum of the players payoffs is minimal for this profile. The sum of players payoffs is maximal for the profile (I, I, I) . The strategy profile in the repeated game is a function $s: S \rightarrow \{I, NI\}^3$, where $S \subset R^3$ is a convex hull of the set of the payoffs in the stage game.

An example of a game considered in the paper is given in Example 2 (section 3). The strategy profile (NI, NI, NI) is the only Nash equilibrium. The common payoff corresponding to the equilibrium profile ($\Sigma = 60$) is the lowest possible one. The strategy NI dominates the strategy I , i.e. the action NI gives higher payoff than the action I despite of the action of other players. So, the example has properties typical for real-life situations called tragedy of commons. The rational player should choose the action NI that dominates I but in real-life situations the cooperation is often observed (comp. Axelrod, 1984). So, it appears a question "How to explain theoretically a player inclination to cooperation that is observed practically?" It is known that one of the strongest factors that motivate cooperative behavior is the repetition of the game. In the paper we assume that the game is repeated infinitely times. Infinite time horizon well approximate real life situations of finite (≥ 20) but unknown time horizons. Our aim is to construct an equilibrium strategy profile in the repeated game that motivates every player to cooperation. We assume that after every repetition players know the average payoff of every player from previous stages. Briefly speaking, an equilibrium strategy of player i bases on the comparison of her average payoff x_i with average payoffs x_j , x_k of remaining players. Player i cooperates (chooses I) if $x_j < x_i + \varepsilon$ and $x_k < x_i + \varepsilon$. If one of the remaining players' average payoff is greater to $x_i + \varepsilon$ then she stops cooperation and chooses NI . Precise definition of an ε -good strategy is given in (19). The positive constant ε is a measure of player's tolerance for others players defection.

Our aim is to construct a strategy profile s^* which is an approximated strong Nash equilibrium in the repeated game under consideration. The constructed equilibrium is safe in the meaning that the payoff of a player choosing strategy s_i^* is not less than the equilibrium payoff in the stage game. This payoff is assured even if the other two players choose an arbitrary strategy. Furthermore, the ε -good strategy guarantees that, in long time horizon, other player's average payoff will not exceed the good strategy player's average payoff by more than ε .

⁶ From now on, we choose to name strategies with I and NI , where I means invest and it corresponds to strategy C and NI corresponds to strategy D .

In the framework of the repeated game, a ε -good strategy is an individually rational strategy. It theoretically explains the players' inclination to cooperation in the repeated three players Prisoner's Dilemma.

The notion of the strong equilibrium in the framework of repeated games was introduced by Aumann (1959, 1961, 1967), who showed that every payoff that belongs to the β -core of the stage game is a strong equilibrium payoff in the corresponding repeated game (comp. Sorin, 1992, Thm. 6.2.2). Despite the fact that the payoff corresponding to the profile (I, I, I) belongs to the β -core, our result is not exactly a case of the Aumann results. We have dropped the assumption of the full monitoring. Players do not observe the full history, i.e. the sequence of actions selected by all players in the previous periods. Instead, we assume that they observe the aggregate history, i.e. the arithmetic mean of the previous payoffs of all the players. It is worth noting that the results on the existence of strong equilibria (comp. Konishi, 1997 and Nessah, 2014) do not apply to the repeated game considered in the present paper.

The repeated Prisoner's Dilemma for more than two players has been considered in Behrstock (2015). The ε -good strategies constructed in the paper have some additional properties to the strategies in Behrstock (2015), in which the authors base on similar approachability results as we do in this paper. The difference is that authors consider N -players Prisoner's Dilemma Game in which strategies are stochastic processes. In our approach all strategies are deterministic.

The paper is organized as follows. In section 2 we present the basic information about sequences related to a map of a convex set. We adopt Blackwell's approachability method (comp. Blackwell, 1956) which was originally used in the framework of 2-player repeated games with vector payoffs. We show that the Blackwell condition is sufficient to obtain the convergence of the sequence of arithmetic means to a set called a weak attractor. The weak attractors introduced in subsection 2.1 have different properties in comparison with approachable sets in the sense of Blackwell. We provide an example of a singleton being a weak attractor that does not satisfy the Blackwell condition. Such a situation is not possible for approachable sets (comp. Shani, 2014, Thm. 8). In repeated games, there is considered a sequence of vector payoffs. Each payoff corresponds to one repetition of the state game. Subsection 2.1 provides us necessary results to analyze the directions in which the trajectory shifts and to examine the convergence of such sequence. This is crucial for defining the payoff in the repeated game. Subsection 2.2 provides basic properties of the Banach limit which shall be used to prove that ε -good strategies are ε Nash equilibria. In some of our arguments we not only require that the sequence of mean payoffs converges to a set, but that almost all its entries belong to the set. A similar problem named strong approachability was considered in Shani (2014). In section 2.3 we adopt a Lyapunov function method for discrete and discontinuous dynamical systems to obtain a deterministic strong approachability result.

In section 3 we consider a repeated 3-player symmetric game. Every player has two actions: invest (I) or not invest (NI). The vector payoff $B = (p_3, p_3, p_3)$ corresponding to the strategy profile (I, I, I) is Pareto optimal and the strategy profile (NI, NI, NI) is a Nash equilibrium in the stage game with the payoff vector (r_0, r_0, r_0) . We assume that, in the repeated game, every player knows the average vector payoff from the previous stages of the game. The strategy $s_i: S \rightarrow \{I, NI\}$, $i \in \{1, 2, 3\}$, is a function from the convex hull S of vector payoffs in the stage game to the set of his actions $\{I, NI\}$. The strategy profile $s = (s_1, s_2, s_3)$ and the vector payoff function $G: \{I, NI\}^3 \rightarrow \mathbb{R}^3$ determine the function $\varphi = G \circ s: S \rightarrow S$. The strategy profile s and the initial point $x_1 \in S$ determine the trajectory $\bar{x}_n(s, x_1)$ of a dynamic system given by

$$\bar{x}_{n+1} = \frac{n\bar{x}_n + \varphi(\bar{x}_n)}{n+1}.$$

Our aim is to construct a strategy profile $s_\varepsilon^* = (s_1^*, s_2^*, s_3^*)$ such that for every $x_1 \in S$

$$\lim_{n \rightarrow \infty} \bar{x}_n = B, \tag{1}$$

where $\bar{x}_n = \bar{x}_n(s_\varepsilon^*, x_1)$. If one player (for example player 3) deviates then

$$\limsup_{n \rightarrow \infty} \bar{x}_n^3 \leq p_3 + \varepsilon, \tag{2}$$

where $\bar{x}_n = \bar{x}_n((s_1^*, s_2^*, s_3), x_1)$ and $s_3: S \rightarrow \{I, NI\}$ is an arbitrary strategy of player 3. If two players deviate (for example players 2 and 3) then

$$\limsup_{n \rightarrow \infty} (\bar{x}_n^2 + \bar{x}_n^3) \leq 2p_3, \tag{3}$$

$$\liminf_{n \rightarrow \infty} \bar{x}_n^1 \geq r_0, \tag{4}$$

$$\lim_{n \rightarrow \infty} \text{dist}(\bar{x}_n, \{x \in S; x_2 \leq x_1 + \varepsilon, x_3 \leq x_1 + \varepsilon\}) = 0, \tag{5}$$

where $\bar{x}_n = \bar{x}_n((s_1^*, s_2, s_3), x_1)$ and $s_2, s_3: S \rightarrow \{I, NI\}$ are the arbitrary strategies of players 2 and 3, respectively. By $\text{dist}(x, A)$ we denote the distance from the point x to the set A , i.e. $\text{dist}(x, A) = \inf\{|x - a|; a \in A\}$.

If the payoff is a Banach limit (comp. Conway, 1985) of the sequence of average payoffs then the strategy profile s_ε^* is a strong ε -Nash equilibrium in the repeated game as a consequence of (1–3). Property (4) implies that the non-deviating player’s payoff is no smaller than the payoff corresponding to the Nash equilibrium in the stage game. Property (5) guarantees that the deviating player’s payoff will not exceed the good strategy player’s payoff by more than ε . The results presented in Theorems 3.1, 3.2, 3.3 give a partial answer to the question asked by Smale in the last Remark in section 1 of Smale (1980, p. 1623).

Section 4 contains concluding remarks. In Appendix we present the proofs of theorems from subsection 2.3.

2. PRELIMINARIES

2.1. Approachability results

Let H be a finite dimensional vector space and $\langle \cdot, \cdot \rangle$, $|\cdot|$ denote an inner product and a norm in H , respectively. We assume that S is a nonempty convex closed subset of H . By $N^\varepsilon(B)$ we denote an ε -neighbourhood of the set B in S , i.e. $N^\varepsilon(B) = \{x \in S: \text{dist}(x, B) < \varepsilon\}$. The closure (the convex hull) of the set A we denote by $\text{cl}(A)$ ($\text{co}(A)$).

We study limit properties of sequences $(\bar{x}_n)_{n=1}^\infty$ defined by a map $\varphi: S \rightarrow S$ and an initial point $x_1 \in S$ by

$$\bar{x}_{n+1} = \frac{n\bar{x}_n + \varphi(\bar{x}_n)}{n+1}, \quad \bar{x}_1 = x_1, \quad (6)$$

The sequence (\bar{x}_n) can be interpreted as a sequence of arithmetic means $\bar{x}_n = \frac{1}{n}(x_1 + \dots + x_n)$, where $x_{k+1} = \varphi(\bar{x}_k)$. The map φ defines a dynamical system $\beta_n: S \rightarrow S$ by

$$\beta_n(x) = \frac{nx + \varphi(x)}{n+1}, \quad n = 1, 2, \dots$$

We denote by $\bar{x}_n(\varphi, x_1)$ a trajectory determined by (6).

We say that a closed set $A \subset S$ is a weak attractor for a dynamic system determined by the map φ if for every $x_1 \in S$ we have

$$\lim_{n \rightarrow \infty} \text{dist}(\bar{x}_n(\varphi, x_1), A) = 0,$$

where $\text{dist}(\cdot, A)$ denotes the distance to the set A . We provide some sufficient conditions for being a weak attractor.

First we formulate Blackwell approachability type theorem that originally was presented in Blackwell (1956) in the framework of repeated games with vector payoffs. We say that a map $\varphi: S \rightarrow S$ satisfies the Blackwell condition for a set $A \subset S$ in the domain $D \subset S$ if

$$\forall x \in D, \exists y \in \Pi_A(x), \quad \langle x - y, \varphi(x) - y \rangle \leq 0, \quad (7)$$

where $\Pi_A(x)$ denote the set of points in A that are proximal to x , i.e. $\Pi_A(x) = \{a \in A: |a - x| = \text{dist}(x, A)\}$.

The deterministic version of the Blackwell approachability result can be formulated in the following way.

Proposition 2.1 Suppose that the map $\varphi: S \rightarrow S$ satisfies the Blackwell condition for a closed set $A \subset S$ in the domain $D \subset S$. If almost all elements of the bounded sequence $\bar{x}_n(\varphi, x_1)$ belong to the set D then

$$\lim_{n \rightarrow \infty} \text{dist}(\bar{x}_n, A) = 0.$$

We provide the proof of Proposition 2.1 for the reader convenience.

Proof: For a sufficiently large n we choose $y \in \Pi_K(\bar{x}_n)$ and then

$$\begin{aligned} \text{dist}(\bar{x}_{n+1}, A)^2 &\leq |\bar{x}_{n+1} - y|^2 = \left| \frac{n}{n+1}(\bar{x}_n - y) + \frac{1}{n+1}(\varphi(x_n) - y) \right|^2 = \\ &= \left(\frac{n}{n+1} \right)^2 |\bar{x}_n - y|^2 + \left(\frac{1}{n+1} \right)^2 |\varphi(x_n) - y|^2 + 2 \frac{n}{(n+1)^2} \langle \bar{x}_n - y, \varphi(x_n) - y \rangle \leq \\ &\leq \left(\frac{n}{n+1} \right)^2 \text{dist}(\bar{x}_n, A)^2 + \left(\frac{1}{n+1} \right)^2 C, \end{aligned}$$

where C is an upper bound of $\text{dist}(x_n, A)$. Setting $d_n = n^2 \text{dist}(\bar{x}_n, A)^2$ we have $d_{n+1} \leq d_n + C$ for $n \geq n_0$. Thus $d_n \leq d_{n_0} + (n - n_0)C$. So

$$\text{dist}(\bar{x}_n, A)^2 \leq \frac{1}{n} \left(d_{n_0} + \frac{n-n_0}{n} C \right). \quad \text{QED}$$

Corollary 2.2 If the map $\varphi: S \rightarrow S$ satisfies the Blackwell condition for a closed set $A \subset S$ in the domain S , then the set A is a weak attractor for φ . If the set $A \subset S$ is convex and the map $\varphi: S \rightarrow A$ maps into the set A then the set A is a weak attractor for φ .

Taking $A = (-\infty, c]$ in Proposition 2.1 we obtain the following property of real sequences.

Corollary 2.3 Suppose that $(a_n)_{n=1}^\infty$ is a bounded sequence in \mathbb{R} and $(\bar{a}_n)_{n=1}^\infty$ is the sequence of arithmetic means, i.e. $\bar{a}_n = \frac{1}{n} \sum_{k=1}^n a_k$. If we have

$$(\bar{a}_n > c \Rightarrow a_{n+1} \leq c)$$

for almost all n and a fixed constant $c \in \mathbb{R}$, then

$$\limsup_{n \rightarrow \infty} \bar{a}_n \leq c.$$

In many cases the set A is a weak attractor despite that the Blackwell condition is not satisfied. Such a situation occurs in repeated games that we study in section 3. Below we present two properties of weak attractors which are necessary for our reasoning.

Proposition 2.4 Suppose that the sets $A, B \subset \mathbb{R}^d$ are nonempty closed and B is bounded. If a sequence x_n satisfies

$$\lim_{n \rightarrow \infty} \text{dist}(x_n, A) = \lim_{n \rightarrow \infty} \text{dist}(x_n, B) = 0,$$

then

$$\lim_{n \rightarrow \infty} \text{dist}(x_n, A \cap B) = 0.$$

Proof: We choose $a_n \in A$, $b_n \in B$ such that

$$|x_n - a_n| = \text{dist}(x_n, A), \quad |x_n - b_n| = \text{dist}(x_n, B)$$

Since the set B is compact, we obtain that the sequences (a_n) , (b_n) , (x_n) are bounded and they have the same nonempty set C of accumulating points. Thus $\lim_{n \rightarrow \infty} \text{dist}(x_n, C) = 0$ and $C \subset A \cap B$.

QED

Proposition 2.5 We suppose that a closed set $A \subset S$ is a weak attractor for the map $\varphi: S \rightarrow S$ and a closed subset $B \subset A$ satisfies

$$\forall \varepsilon > 0, \exists \delta > 0, \quad \varphi \text{ satisfies the Blackwell condition} \\ \text{for the set } \text{cl}(N^\varepsilon(B)) \cap A \text{ in the domain } N^\delta(A). \quad (8)$$

Then the set B is a weak attractor for φ .

Proof: Fix $x_1 \in S$ and $\varepsilon > 0$. By (8), we choose $\delta > 0$ such that almost all elements of the trajectory $\bar{x}_n(\varphi, x_1)$ belongs to $N^\delta(A)$. By Proposition 2.1, we obtain

$$\lim_{n \rightarrow \infty} \text{dist}(\bar{x}_n, \text{cl}(N^\varepsilon(B)) \cap A) = 0.$$

Thus

$$\limsup_{n \rightarrow \infty} \text{dist}(\bar{x}_n, B) \leq \varepsilon. \quad \text{QED}$$

The method illustrated in Proposition 2.4 and Proposition 2.5 bases on the scheme that we explain in the following example.

Example 1 Let $S = \mathbb{R}^2$, $a, b \in \mathbb{R}^2$, $a_2 < 0$, $b_2 > 0$, $a_1 \neq b_1$ and

$$\varphi(x, y) = \begin{cases} a & \text{if } y > 0, \\ b & \text{if } y \leq 0. \end{cases}$$

We show that $\lim_{n \rightarrow \infty} \bar{x}_n = d$ for every $\bar{x}_1 \in \mathbb{R}^2$, where the limit d is the point of intersection of the interval \overline{ab} with the line $p = \{(x, y) : y = 0\}$. The set $D = \{d\}$ does not satisfy condition (7). Indeed, if $a_1 < b_1$ and $x > d_1$ then $\varphi(x, 0) = b$ and $\langle (x, 0) - (d_1, d_2), \varphi(x, 0) - (d_1, d_2) \rangle > 0$. To show that the set D is a weak attractor we point out weak attractors A, B such that $D = A \cap B$. We set $A = p$ and $B = \overline{ab}$. The sets A, B satisfy the Blackwell condition (7). By Theorem 2.1, we have

$$\lim_{n \rightarrow \infty} \text{dist}(\bar{x}_n, A) = \lim_{n \rightarrow \infty} \text{dist}(\bar{x}_n, B) = 0.$$

Applying Proposition 2.4 we obtain that $\lim_{n \rightarrow \infty} \bar{x}_n = d$.

Finally, we shall formulate a property of the dynamical system.

Proposition 2.6 If the set S is bounded then for every $\xi > 0$ there exists $N \in \mathbb{N}$ such that for all $n > N$ and for all $x \in S$

$$|\beta_n(x) - x| < \xi,$$

where the map $\varphi : S \rightarrow S$ determining β_n is arbitrary.

2.2. Payoff in the repeated game

Considering a sequence of payoffs in the repeated games we always receive a bounded sequence. As we presented in (6), the dynamic is the vector of the arithmetic mean of the payoffs received in the previous repetitions. To analyze such sequence, the following proposition shall be useful.

Proposition 2.7 Suppose that $a_0, a_1, \dots, a_k \in \mathbb{R}^d$ and let $T \in \mathbb{N}$. Then for all $\varepsilon > 0$ and for all $n_1, \dots, n_k \geq 0$ such that $n_1 + \dots + n_k = n$, where n is sufficiently large, we have

$$\frac{T}{T+n} a_0 + \frac{n_1}{T+n} a_1 + \dots + \frac{n_k}{T+n} a_k \in N_\varepsilon(\text{co}\{a_1, \dots, a_k\}).$$

Proposition 2.7 is a consequence of the fact that $\frac{T}{T+n} a_1 + \frac{n_1}{T+n} a_1 + \dots + \frac{n_k}{T+n} a_k \in \text{co}\{a_1, \dots, a_k\}$ and $\frac{T}{T+n} |a_0 - a_1|$ is small where n is sufficiently large.

To define the payoff in repeated games we shall use the Banach limit (comp. Conway, 1985). The Banach limit L is a continuous linear functional definite on the space l^∞ of bounded scalar sequences. If (x_n) is a bounded sequence of points in \mathbb{R}^d then $\text{Lim}(x_n) := (\text{Lim}(x_{n_1}), \text{Lim}(x_{n_2}), \dots, \text{Lim}(x_{n_d}))$, where $x_n = (x_{n_1}, x_{n_2}, \dots, x_{n_d})$. So Banach Limit can be extended onto the space of bounded

sequences of points in \mathbb{R}^d . If $\varphi: \mathbb{R}^d \rightarrow \mathbb{R}$ is a linear functional then $\varphi(\text{Lim}(x_n)) = \text{Lim}(\varphi(x_n))$.

Proposition 2.8 If A is a compact convex subset of \mathbb{R}^d and a sequence $(x_n) \subset \mathbb{R}^d$ satisfies $\lim_{n \rightarrow \infty} \text{dist}(x_n, A) = 0$, then $\text{Lim}(x_n) \in A$.

Proof. Suppose to the contrary that $\text{Lim}(x_n) \notin A$. Then there exists a functional $\varphi: \mathbb{R}^d \rightarrow \mathbb{R}$ such that $\varphi(\text{Lim}(x_n)) > \sup_{a \in A} \varphi(a)$. We have $\limsup_{n \rightarrow \infty} \varphi(x_n) \leq \sup_{a \in A} \varphi(a)$. Thus

$$\varphi(\text{Lim}(x_n)) = \text{Lim}(\varphi(x_n)) \leq \limsup_{n \rightarrow \infty} \varphi(x_n) \leq \sup_{a \in A} \varphi(a)$$

which gives the contradiction. QED

2.3. A lapunov type results

The Lapunov function method is typically used to study stability of equilibrium points for dynamical systems. Using the Lapunov function method we obtain a strong approachability result for a dynamical system determined by a multivalued map.

Let H be a Hilbert space and $p_1, \dots, p_k \in H$ be unit vectors, i.e. $\|p_i\| = 1$. We define a function $V: H \rightarrow \mathbb{R}$ by

$$V(x) = \max_{i \in \{1, \dots, k\}} V_i(x) \quad \text{where } V_i(x) = \langle p_i, x \rangle. \quad (9)$$

The function V is a support function of the set $\{v_1, \dots, v_k\}$. So, the function V is convex, positively homogeneous and lipschitz continuous with the constant $L = 1$ (see [11]).

Set

$$\Delta_c = \bigcap_{i=1}^k \{x \in H: V_i(x) < c\} = \{x \in H: V(x) < c\}.$$

Let us denote by $\varphi: S \rightarrow S$ a multivalued map of a subset $S \subset H$.

Definition 2.9 We say that V is the Lapunov type function for the multivalued map φ with the constant $c > 0$ if

$$\exists 0 < \delta < c, \forall x \in S \setminus \Delta_c, \forall i = 1, \dots, k, \forall \omega \in \varphi(x), (V_i(x) \geq V(x) - \delta \Rightarrow V_i(\omega) \leq 0). \quad (10)$$

If the function V satisfies

$$\forall x \in S \setminus \Delta_c, \forall \omega \in \varphi(x), \forall i \in \{1, \dots, k\} \quad V_i(x) > 0 \Rightarrow V_i(\omega) \leq 0, \quad (11)$$

then V is the Lapunov type function for φ with the constant c .

If V is the Lapunov type function for φ with the constant c and $c_1 > c$ then V is the Lapunov type function for φ with the constant c_1 .

To explain why we say that V is the Lapunov type function observe that if $V_i(x) = V(x)$ then $p_i \in \partial V(x)$, where $\partial V(x)$ is the subdifferential of a convex function. The condition (10) implies the following inequality

$$\langle p_i, \omega - x \rangle \leq \langle p_i, \omega \rangle - \langle p_i, x \rangle \leq 0 - V(x) + \delta < \delta - c < 0 \quad \text{for } x \in S \setminus \Delta_c,$$

which means that V is the Lapunov function for the vector field $f(x) = \omega - x$.

Proposition 2.10 Let S be a nonempty bounded convex subset of H and the function $V: H \rightarrow \mathbb{R}$ given by (9) be the Lapunov type function for the multivalued map $\varphi: S \rightarrow S$ with the constant $c > 0$. If a sequence $(\bar{x}_n)_{n=1}^\infty$ satisfies

$$\bar{x}_1 = x_1 \in S, \quad \bar{x}_{n+1} = \frac{n\bar{x}_n + x_{n+1}}{n+1}, \quad x_{n+1} \in \varphi(\bar{x}_n), \tag{12}$$

then

$$\forall c_1 > c, \exists N, \forall n \geq N, \quad \bar{x}_n \in \Delta_{c_1}.$$

The proof of Proposition 2.10 is technical and it is presented in Appendix.

3. THE MODEL AND MAIN RESULTS

Let G be a 3-player symmetric game and every player has two pure actions: "invest" (I) or "not invest" (NI). By P_i (P_{NI}) we denote the payoff for an investing (not investing) player. All payoffs depend on the total number of investing players. If $n \in \{0, 1, 2, 3\}$ is the total number of investing players, then

n	$P_i(n)$	$P_{NI}(n)$
0	–	r_0
1	p_1	r_1
2	p_2	r_2
3	p_3	–

The game G in the normal form is given by the matrix:

I	(p_2, r_2, p_2)	(p_3, p_3, p_3)
NI	(r_1, r_1, p_1)	(r_2, p_2, p_2)
NI		I

when the third player invests, and by the matrix

I	(p_1, r_1, r_1)	(p_2, p_2, r_2)
NI	(r_0, r_0, r_0)	(r_1, p_1, r_1)
NI		I

when the third player does not invest.

We shall assume that the functions $P_1(\cdot)$, $P_{NI}(\cdot)$ are increasing:

$$0 < r_0 < r_1 < r_2 \quad \text{and} \quad 0 < p_1 < p_2 < p_3. \quad (13)$$

We assume that

$$p_1 < r_0. \quad (14)$$

By (14), the outcome (NI, NI, NI) is a Nash equilibrium. We assume that the more players invest, the greater the sum of all players payoffs is, i.e.

$$3r_0 < p_1 + 2r_1 < 2p_2 + r_2 < 3p_3. \quad (15)$$

By (15), the vector payoff (p_3, p_3, p_3) is Pareto optimal. In fact, the condition (15) means even more – the vector payoff (p_3, p_3, p_3) maximize the sum of payoffs. To obtain a strong equilibrium in the repeated game we assume that:

$$p_1 + r_1 < 2p_3. \quad (16)$$

We additionally assume that:

$$p_2 < r_2. \quad (17)$$

Observe that from the opposite inequality $r_2 \leq p_2$ implies that (p_3, p_3, p_3) is a Nash equilibrium payoff, what we wanted to avoid.

We introduce the following notations

$$\begin{aligned} A &= (r_0, r_0, r_0), \\ B &= (p_3, p_3, p_3), \\ C_1^1 &= (p_1, r_1, r_1), \\ C_2^1 &= (r_1, p_1, r_1), \\ C_3^1 &= (r_1, r_1, p_1), \\ C_1^2 &= (r_2, p_2, p_2), \\ C_2^2 &= (p_2, r_2, p_2), \\ C_3^2 &= (p_2, p_2, r_2). \end{aligned}$$

If i players invest ($i \in \{1, 2\}$) then C_j^i denotes the vector payoff in the game G . If $i = 1$ then j shows which one invests, while if $i = 2$ then j tells which player does not invest.

The strategy profile in the iterated game is given by a map $s: S \rightarrow \{I, NI\}^3$, where S is the convex hull of vector payoffs set, i.e.

$$S = \text{co}\{A, B, C_1^1, C_2^1, C_3^1, C_1^2, C_2^2, C_3^2\}.$$

The strategy profile s determines a dynamical process $\beta_n: S \rightarrow S$

$$\beta_n(x) = \frac{nx + \varphi(x)}{n+1}, \quad \text{for } x \in S, \quad n \in \mathbb{N}, \tag{18}$$

where $\varphi: S \rightarrow S$ is given by the formula $\varphi = G \circ s$. Observe that a pair (s, x_1) , where s is a strategy profile and $x_1 \in S$, uniquely determines a sequence $(\bar{x}_n)_{n=1}^\infty$ by:

$$\bar{x}_1 = x_1, \quad \bar{x}_{n+1} = \beta_n(\bar{x}_n).$$

We denote the obtained sequence by $\bar{x}_n(s, x_1)$. A similar construction of a sequence was considered in section 2. The strategy profile s and the initial point $x_1 \in S$ uniquely determine a play path. The action profile in the next stage $s(\bar{x}_n)$ depends on the average vector payoff \bar{x}_n . The element x_{n+1} is the vector payoff in $n + 1$ stage. We do not assume that the players observe the full history of the game. Instead, they observe aggregated history – the arithmetic mean of vector payoffs.

Motivated by the Smale construction in Smale (1980) we define an ε -good strategy for the i -th player $s_i^\varepsilon: S \rightarrow \{I, NI\}$ by

$$s_i^\varepsilon(x) = \begin{cases} I & \text{if } x \in V_i, \\ NI & \text{if } x \in S \setminus V_i, \end{cases} \tag{19}$$

where

$$\begin{aligned} V_i &= \Omega_i^\varepsilon \setminus W_i, \\ \Omega_i^\varepsilon &= \{x \in S: x_i > x_j - \varepsilon \text{ and } x_i > x_k - \varepsilon\}, \\ W_i &= \{x \in S: x_i < r_0 \text{ or } x_j + x_k > 2p_3\}, \end{aligned}$$

where i, j, k are pairwise different elements of the set of players $\{1, 2, 3\}$. The player invests if his average payoff is greater than the every other players' average payoff minus ε . The player stops investing if his playing I has been exploited by his opponents, that is either the average payoff of the player is lower than the payoff guaranteed by Nash equilibrium ($x_i < r_0$) or the sum of the other players' average payoffs is greater then the sum of their payoffs corresponding to the Pareto optimal profile (I, I, I) ($x_j + x_k > 2p_3$).

First we consider the case when all players choose good strategies. Then the average payoffs vector tends to the point B corresponding to the Pareto optimal profile (I, I, I) .

Theorem 3.1 Suppose that $s_i^\varepsilon: S \rightarrow \{I, NI\}$ are the ε -good strategies for $i = 1, 2, 3$. Then

$$\lim_{T \rightarrow \infty} \bar{x}_T = B,$$

where $\bar{x}_T = \bar{x}_T((s_1^\varepsilon, s_2^\varepsilon, s_3^\varepsilon), x_1)$ and x_1 is an arbitrary element of S .

Now, we consider the case when two players play good strategies and the third one deviates and chooses an arbitrary strategy. The deviating player does not improve their payoff more than $\frac{2}{3}\varepsilon$, where the positive constant ε can be chosen arbitrarily small by the two non-deviating players.

Theorem 3.2 Suppose that the first and the second player choose the ε -good strategies $s_1^\varepsilon, s_2^\varepsilon$ and the third player plays an arbitrary strategy $s_3: S \rightarrow \{I, NI\}$. Then

$$\limsup_{T \rightarrow \infty} \bar{x}_T^3 \leq p_3 + \varepsilon \frac{2}{3}, \quad (20)$$

where $\bar{x}_T = \bar{x}_T((s_1^\varepsilon, s_2^\varepsilon, s_3), x_1)$ and x_1 is an arbitrary element of S .

At the end of the section we show an example of the third player strategy, such that the upper limit of his average payoffs is strictly greater than p_3 .

Now, we consider the case when two players deviate.

Theorem 3.3 Suppose s_1^ε is the ε -good strategy for the first player and s_2, s_3 are arbitrary strategies. Then

$$\liminf_{T \rightarrow \infty} \bar{x}_T^1 \geq r_0, \quad (21)$$

$$\limsup_{T \rightarrow \infty} (\bar{x}_T^2 + \bar{x}_T^3) \leq 2p_3, \quad (22)$$

$$\lim_{T \rightarrow \infty} \text{dist}(\bar{x}_T, V_1) = 0, \quad (23)$$

where $\bar{x}_T = \bar{x}_T((s_1^\varepsilon, s_2, s_3), x_1)$ and x_1 is an arbitrary element of S .

Suppose that the payoff in the repeated game is defined as the Banach limit of average payoffs. The inequality (22) provides that if two players deviate then at least one of them will not improve his payoff. Conclusions (21) and (23) mean that the good strategy is safe, i.e. the non-deviating player's payoff is not smaller than the Nash equilibrium payoff in the stage game and, moreover, the deviating player's payoff is not greater than the non-deviating player's payoff plus ε (comp. Proposition 2.8).

By Theorems 3.1 – 3.3, we obtain

Corollary 3.4 The strategy profile $s^\varepsilon = (s_1^\varepsilon, s_2^\varepsilon, s_3^\varepsilon)$ satisfies (1-5). If we define the payoff in the repeated game as a Banach limit of average payoffs, i.e. $\text{Lim} \bar{x}_T$ then the strategy profile s^ε is a safe and strong ε Nash equilibrium.

Below we provide some elementary properties of sets V_i that are used in the definition of good strategies. We assume that i, j, k are pairwise different elements of the set of players $\{1, 2, 3\}$. We shall use the following notations

$$\begin{aligned} V^3 &= \bigcap_{i=1}^3 V_i \\ V_i^2 &= (S \setminus V_i) \cap V_j \cap V_k \\ V_i^1 &= V_i \cap (S \setminus V_j) \cap (S \setminus V_k) \end{aligned}$$

If each player plays good strategy then

$$\varphi(x) = \begin{cases} B & \text{if } x \in V^3 \\ C_i^1 & \text{if } x \in V_i^1 \text{ for } i \in \{1, 2, 3\}. \\ C_i^2 & \text{if } x \in V_i^2 \text{ for } i \in \{1, 2, 3\} \end{cases} \quad (24)$$

Proposition 3.5 Suppose that player i plays ε_i -good strategy for $i = 1, 2, 3$. Then

$$\Omega_i \cap W_i = \emptyset, \quad (25)$$

$$V_i^1 \subset \Omega_i, \quad (26)$$

where

$$\Omega_i = \{x \in S: x_i = \max\{x_1, x_2, x_3\}\}.$$

If we assume that $\varepsilon_i = \varepsilon_j (=:\varepsilon)$ then

$$V_i \cap (S \setminus V_j) \subset \Omega_i \cup \Phi_j. \quad (27)$$

If we assume that $\varepsilon_i = \varepsilon_j = \varepsilon_k (=:\varepsilon)$ then for every $i \in \{1, 2, 3\}$ we have

$$V_i^2 \subset \Phi_i, \quad (28)$$

where

$$\Phi_i = \{x \in S: x_i = \min\{x_1, x_2, x_3\}\}.$$

Proof: If $x_i < r_0$ and $x_i = \max\{x_1, x_2, x_3\}$ then $x_1 + x_2 + x_3 < 3r_0$. If $x_j + x_k > 2p_3$ and $x_i = \max\{x_1, x_2, x_3\}$ then $x_1 + x_2 + x_3 > 3p_3$. Since $3r_0 \leq x_1 + x_2 + x_3 \leq 3p_3$ for $x \in S$, we obtain (25).

As $(S \setminus V_j) \cap \Omega_j = \emptyset$ and $(S \setminus V_k) \cap \Omega_k = \emptyset$ we have $(S \setminus V_j) \cap (S \setminus V_k) \subset S \setminus (\Omega_j \cup \Omega_k) \subset \Omega_i$, and consequently we obtain (26).

To prove conclusion (27) we take $i = 1, j = 2$. As $(W_2 \setminus W_1) \cap \Phi_1 = \emptyset$ and $(\Omega_3 \setminus \Phi_1) \subset \Phi_2$ we obtain $W_2 \setminus W_1 \subset (\Omega_1 \cup \Omega_3) \setminus \Phi_1 \subset \Omega_1 \cup \Phi_2$. If $x \in \Omega_1^{\varepsilon} \setminus \Omega_2^{\varepsilon}$ then either

$$x_2 \leq x_1 - \varepsilon$$

or

$$x_2 \leq x_3 - \varepsilon \text{ and } x_1 > x_3 - \varepsilon \text{ (} x \in \Omega_1^{\varepsilon} \text{)}.$$

In both cases we obtain $x_2 < x_1$ and thus $x \in \Omega_1 \cup \Phi_2$. Since $V_1 \cap (S \setminus V_2) \subset (\Omega_1^{\varepsilon} \setminus \Omega_2^{\varepsilon}) \cup (W_2 \setminus W_1)$, we conclude that

$$V_1 \cap (S \setminus V_2) \subset \Omega_1 \cup \Phi_2.$$

If $x_i \leq x_j - \varepsilon$ ($x \notin \Omega_i^{\varepsilon}$) and $x_k > x_j - \varepsilon$ ($x \in \Omega_k^{\varepsilon}$) then $x \in \Phi_i$. If $x_i \leq x_k - \varepsilon$ ($x \notin \Omega_i^{\varepsilon}$) and $x_j > x_k - \varepsilon$ ($x \in \Omega_j^{\varepsilon}$) then $x \in \Phi_i$. Thus $(S \setminus \Omega_i^{\varepsilon}) \cap \Omega_j^{\varepsilon} \cap \Omega_k^{\varepsilon} \subset \Phi_i$.

If $x \in V_j$ then $x \notin W_j$ and hence $x_j \geq r_0$ and $x_i + x_k \leq 2p_3$. If $x \in W_i \cap V_j \cap V_k$ then either $x_i < r_0, x_j \geq r_0, x_k \geq r_0$ or $x_j + x_k > 2p_3, x_i + x_k \leq 2p_3, x_i + x_j \leq 2p_3$. In both cases we deduce that $x \in \Phi_i$. So $V_i^2 \subset \Phi_i$. QED

First we prove Theorem 3.3.

Proof: The strategy s_1^* is the ε -good strategy, so if $\bar{x}_T^1 < r_0$ then $\bar{x}_T = (\bar{x}_T^1, \bar{x}_T^2, \bar{x}_T^3) \in W_1$ and $s_1^*(\bar{x}_T) = NI$. It means that the next vector payoff x_{T+1} belongs to the set $\{A, C_2^1, C_3^1, C_1^2\}$, so $x_{T+1}^1 \in \{r_0, r_1, r_2\}$, i.e. $x_{T+1}^1 \geq r_0$ (see (13)). By Corollary 2.3 we obtain that $\limsup_{T \rightarrow \infty} -\bar{x}_T^1 \leq -r_0$, so $\liminf_{T \rightarrow \infty} \bar{x}_T^1 \geq r_0$.

Similarly, if $\bar{x}_T^2 + \bar{x}_T^3 > 2p_3$ then $s_1^*(\bar{x}_T) = NI$. Thus the sum $x_{T+1}^2 + x_{T+1}^3$ is one of the numbers: $2r_0, p_1 + r_1, 2p_2$. From the assumptions (13), (15) and (16), it follows that $x_{T+1}^2 + x_{T+1}^3 \leq 2p_3$. By Corollary 2.3, we get

$$\limsup_{T \rightarrow \infty} \bar{x}_T^2 + \bar{x}_T^3 \leq 2p_3.$$

If $x \in S \setminus V_1$ then $s_1^*(\bar{x}_T) = NI$. So, $\varphi(x) = G((s_1^*, s_2, s_3)(x)) \in \{A, C_2^1, C_3^1, C_1^2\} \subset V_1$. By Corollary 2.2, the set V_1 is a weak attractor for φ . QED

Let $\pi_u: \mathbb{R}^3 \rightarrow u$ be the orthogonal projection onto the line $u = \{x \in \mathbb{R}^3: x_1 = x_2 = x_3\}$ and $\pi_P: \mathbb{R}^3 \rightarrow P$ be the orthogonal projection onto the plane $P = \{x \in \mathbb{R}^3: x_1 + x_2 + x_3 = 0\}$. Obviously $\pi_u(x) = \left(\frac{x_1+x_2+x_3}{3}, \frac{x_1+x_2+x_3}{3}, \frac{x_1+x_2+x_3}{3}\right)$ and $\pi_P(x) = x - \pi_u(x)$. In the remainder of the section we denote the projection of a point (a set) A onto the plane P by \tilde{A} , i.e. $\tilde{A} = \pi_P(A)$. The projection of the set S onto the plane P :

$$\tilde{S} = \pi_P(S)$$

is the convex hull of the hexagon with successive vertexes $\tilde{C}_1^1, \tilde{C}_2^2, \tilde{C}_3^3, \tilde{C}_1^2, \tilde{C}_2^1, \tilde{C}_3^2$.

Set

$$\begin{aligned} v_1 &= \frac{1}{\sqrt{2}}(0, -1, 1), & v_4 &= -v_1, \\ v_2 &= \frac{1}{\sqrt{2}}(-1, 0, 1), & v_5 &= -v_2, \\ v_3 &= \frac{1}{\sqrt{2}}(-1, 1, 0), & v_6 &= -v_3, \end{aligned}$$

and

$$\Delta_c(K) = \bigcap_{i \in K} \{y \in \tilde{S} : \langle v_i, y \rangle < c\},$$

where $K \subset \{1, \dots, 6\}$ and $c > 0$. One can easily check that

$$\begin{aligned} x \in \Omega_1^\varepsilon &\Leftrightarrow \pi_P(x) \in \Delta_c(\{2, 3\}), \\ x \in \Omega_2^\varepsilon &\Leftrightarrow \pi_P(x) \in \Delta_c(\{1, 6\}), \\ x \in \Omega_3^\varepsilon &\Leftrightarrow \pi_P(x) \in \Delta_c(\{4, 5\}), \end{aligned}$$

where $c = \frac{\varepsilon}{\sqrt{2}}$. Setting $\Omega^\varepsilon = \bigcap_{i=1}^3 \Omega_i^\varepsilon$ and $\Delta_c = \Delta_c(\{1, \dots, 6\})$ we obtain

$$x \in \Omega^\varepsilon \Leftrightarrow \pi_P(x) \in \Delta_c. \tag{29}$$

Now, we are able to prove Theorem 3.1.

Proof: Fix $x_1 \in S$. It is sufficient to show that in the sequence $\bar{x}_T = \bar{x}_T(s^*, x_1)$ there exists an element \bar{x}_N belonging to V^3 , where $s^* = (s_1^\varepsilon, s_2^\varepsilon, s_3^\varepsilon)$. Indeed, if $\bar{x}_N \in V^3$ then $\bar{x}_{N+k} = \frac{N}{N+k}\bar{x}_N + \frac{k}{N+k}B$, so $\lim_{T \rightarrow \infty} \bar{x}_T = B$.

First we show that almost all elements of the sequence \bar{x}_T belong to $\Omega^\eta = \bigcap_{i=1}^3 \Omega_i^\eta$, for every $\eta > 0$.

The map φ given by (24) is determined by the strategy profile s^* , i.e. $\varphi = G \circ s^*$. Consider $\tilde{\varphi}: \tilde{S} \rightarrow \tilde{S}$ and $V: P \rightarrow \mathbb{R}$ given by

$$\begin{aligned} \tilde{\varphi}(x) &= \{\pi_P(\varphi(y)) : \pi_P(y) = x\}, \\ V(x) &= \max\{\langle v_i, x \rangle : i = 1, \dots, 6\}. \end{aligned}$$

We verify that V is a Lapunov type function for $\tilde{\varphi}$ with the constant c , for an arbitrary $c > 0$. Let us fix $x \in \tilde{S}$ such that $\langle v_1, x \rangle > 0$. If $y \in S$ and $\pi_P(y) = x$ then $\langle v_1, y \rangle = \langle v_1, x \rangle$. Thus $y_3 - y_2 > 0$ and therefore $y \notin \Omega_2 \cup \Phi_3$. By (26), (28), we have $y \notin V_2^1 \cup V_3^2$. Since $\varphi(y) \in \{C_1^1, C_3^1, C_1^2, C_2^2, B\}$, we obtain $\langle v_1, \omega \rangle \leq 0$ for $\omega \in \tilde{\varphi}(x)$. We use similar arguments to show that if $\langle v_i, x \rangle > 0$ and $\omega \in \tilde{\varphi}(x)$ then $\langle v_i, \omega \rangle \leq 0$, for $i = 2, \dots, 6$.

Fix $\eta < \min\{\varepsilon, p_3 - \frac{2p_2+r_2}{3}, \frac{p_1+2r_1}{3} - r_0\}$. By Proposition 2.10 and (29), there exists N such that $\bar{x}_n \in \Omega^\eta$ for $n > N$. We claim that there exists $M > N$ such that $\bar{x}_M \in V^3$. Suppose to the contrary that $\bar{x}_M \notin V^3$ for every $M > N$. Then $\varphi(\bar{x}_M) \in \{C_1^1, C_2^1, C_3^1, C_1^2, C_2^2, C_3^2\}$ for $M > N$. By Proposition 2.7, we obtain that $z_M \in (\frac{p_1+2r_1-\eta}{3}, \frac{2p_2+r_2+\eta}{3})$ for M sufficiently large, where the point (z_M, z_M, z_M) is the projection of \bar{x}_M onto u .

But, if $x \in \Omega^\eta \setminus V^3$ then $\frac{x_1+x_2+x_3}{3} \notin (\frac{p_1+2r_1-\eta}{3}, \frac{2p_2+r_2+\eta}{3})$. Indeed, if $x_i + x_j > 2p_3$ and $x \in \Omega^\eta$ then $\frac{x_1+x_2+x_3}{3} > p_3 - \frac{\eta}{3}$. If $x_i < r_0$ and $x \in \Omega^\eta$ then $\frac{x_1+x_2+x_3}{3} < r_0 + \frac{2}{3}\eta$.

QED

Now, we are in a position to prove Theorem 3.2.

Proof: Let $x_1 \in S$ and $\eta > 0$. Our aim is to prove that almost all elements of the sequence $\bar{x}_T = \bar{x}_T((s_1^\varepsilon, s_2^\varepsilon, s_3), x_1)$ belongs to $\Omega^{\varepsilon+\eta} := \Omega_1^{\varepsilon+\eta} \cap \Omega_2^{\varepsilon+\eta}$. We have

$$x \in \Omega^{\varepsilon+\eta} \Leftrightarrow \pi_p(x) \in \Delta_c(\{1, 2, 3, 6\}), \tag{30}$$

where $c := \frac{\varepsilon+\eta}{\sqrt{2}}$. We show that the function $V^*: P \rightarrow \mathbb{R}$ given by

$$V^*(x) = \max\{v_i, x\}; i = 1, 2, 3, 6\}$$

is the Lapunov type function for $\tilde{\varphi}^*: \tilde{S} \rightarrow \tilde{S}$ with the constant c , where

$$\tilde{\varphi}^* = \{\pi_p(z); z \in \varphi^*(y), \quad \pi_p(y) = x\}$$

and

$$\varphi^*(y) = \begin{cases} \{C_1^1, C_2^2\} & \text{if } y \in V_1 \cap (S \setminus V_2), \\ \{C_2^1, C_1^2\} & \text{if } y \in (S \setminus V_1) \cap V_2, \\ \{B, C_3^2\} & \text{if } y \in V_1 \cap V_2, \\ \{A, C_3^1\} & \text{if } y \in (S \setminus V_1) \cap (S \setminus V_2). \end{cases}$$

The map $\varphi: S \rightarrow S$ induced by the profile $(s_1^\varepsilon, s_2^\varepsilon, s_3)$ is a selection of φ^* .

If $\langle v_6, x \rangle > 0$ ($x \in \tilde{S}$) and $\pi_p(y) = x$ ($y \in S$) then $y_1 > y_2$ and thus $y \notin \Omega_2 \cup \Phi_1$. By (27), we have $V_2 \cap (S \setminus V_1) \subset \Omega_2 \cup \Phi_1$. Thus $\varphi^*(y) \cap \{C_2^1, C_1^2\} = \emptyset$. So, we have $\langle v_6, \omega \rangle < 0$ for $\omega \in \tilde{\varphi}^*(x)$.

Using similar arguments we show that if $\langle v_3, x \rangle > 0$ then $\langle v_3, \omega \rangle \leq 0$ for $\omega \in \tilde{\varphi}^*(x)$.

Suppose that $\langle v_1, x \rangle \geq V^*(x) - \delta$ ($x \in \tilde{S}$) and $\pi(y) = x$ ($y \in S$), where $\delta < \frac{\eta}{\sqrt{2}}$. Then $\langle v_1, x \rangle = \langle v_1, y \rangle > \frac{\varepsilon}{\sqrt{2}}$. If $z \in \Omega_2^\varepsilon$ then $\langle v_1, z \rangle \geq \frac{1}{\sqrt{2}}(z_3 - z_2) > \frac{\varepsilon}{\sqrt{2}}$. Thus, we have $y \notin \Omega_2^\varepsilon \supset V_2$ and therefore $\varphi^*(y) \subset \{C_1^1, C_2^2, C_3^1, A\}$. So, $\langle v_1, \omega \rangle \leq 0$ for $\omega \in \tilde{\varphi}^*(x)$.

In the similar way we prove that if $\langle v_2, x \rangle \geq V^*(x) - \delta$ and $\omega \in \tilde{\varphi}^*(x)$ then $\langle v_2, \omega \rangle \leq 0$.

By Proposition 2.10, we obtain that almost all elements of the sequence $(\pi_P(\bar{x}_T))$ belongs to $\Delta_c(\{1, 2, 3, 6\})$. By (30), we have that almost all elements of the sequence (\bar{x}_T) belongs to $\Omega^{\varepsilon+\eta}$. If $x \in \Omega^{\varepsilon+\eta}$ then $x_1 > x_3 - (\varepsilon + \eta)$ and $x_2 > x_3 - (\varepsilon + \eta)$ and so $x_3 < p_3 + \frac{2}{3}(\varepsilon + \eta)$ ($x_1 + x_2 + x_3 \leq 3p_3$ for $x \in S$).

QED

Remark. Reasoning as in the proofs of Theorem 3.2 and Theorem 3.3, we can conclude that good strategies are safe and strong Nash equilibria not only in the class of Smale's strategies, but also if "loyal" players adopt good strategies, then "disloyal" players can even play the random choice in each repetition. It does not change the properties (20), (21), (22) and (23).

Example 2 Let the stage game G be given by:

n	$P_1(n)$	$P_{NI}(n)$
0	—	20
1	10	28
2	18	36
3	26	—

This game satisfies conditions (13) – (17).

Let $s_i^*: S \rightarrow \{I, NI\}$ be the ε -good strategy for the i -th player, $i = 1, 2$, and $0 < \varepsilon < \frac{1}{2}$. Let $Z = \text{conv}\{A, B, C_3^1, C_3^2\} = \{x \in S: x_1 = x_2\}$ and $D = (26 - \frac{\varepsilon}{2}, 26 - \frac{\varepsilon}{2}, 26 + \frac{\varepsilon}{2})$. We present the construction of the third player strategy $s_3^*: S \rightarrow \{I, NI\}$ such that

$$\lim_{T \rightarrow \infty} x_T(s^*, x_1) = D \text{ for every } x_1 \in Z.$$

We have $V_1 \cap Z = V_2 \cap Z$. We set

$$s_3^*(x) = \begin{cases} NI & \text{if } x \in V_1 \cap Z \cap \text{co}\{B, D, C_3^1\} \\ I & \text{elsewhere.} \end{cases}$$

The map φ induced by the strategy profile $s^* = (s_1^*, s_2^*, s_3^*)$ is given by

$$\varphi(x) = \begin{cases} B & \text{if } x \in V_1 \cap Z \setminus \text{co}\{B, D, C_3^1\}, \\ C_3^2 & \text{if } x \in V_1 \cap Z \cap \text{co}\{B, D, C_3^1\}, \\ C_3^1 & \text{if } x \in Z \setminus V_1. \end{cases}$$

The values of the map φ outside the set Z have no influence onto the trajectory $\bar{x}_T(s^*, x_1)$ if $x_1 \in Z$. The map $\varphi: Z \rightarrow Z$ satisfies the Blackwell condition for the triangle $\text{co}\{C_3^1, C_3^2, D\}$ in the domain Z . The map $\varphi: Z \rightarrow Z$ satisfies the Blackwell condition for the sum of intervals $\overline{BD} \cup \overline{DC}_3^1$ in the domain Z . By Proposition 2.1, the sets $\text{co}\{C_3^1, C_3^2, D\}$ and $\overline{BD} \cup \overline{DC}_3^1$ are weak attractors. To conclude that the interval \overline{BD} is a weak attractor we apply Proposition 2.5 taking $A = \overline{BD} \cup \overline{DC}_3^1$ and $B = \overline{BD}$. By Proposition 2.4, the intersection of weak attractors $\text{co}\{C_3^1, C_3^2, D\}$ and $A = \overline{BD}$ is a weak attractor. The intersection equals to the set $\{D\}$.

4. CONCLUSIONS

This paper is concerned with the specific model of social dilemmas. Such models have a very special place in game theory as they describe real social problems of modern world: resources depletion, pollution and overpopulation. The main characteristic of such models is that each player gain more by not cooperating when opponents fix their choices and all individuals are better off if all cooperate. The lack of optimality of Nash equilibrium is the most interesting problem, because as we can observe in the real world, people are keen to cooperate with each other on the certain conditions. As we can find in Axelrod (1984), strategies that effectively encourage people to cooperate are: nice, forgiving, retaliatory and are found on simple rules.

The key idea in our approach is to apply Smale's idea for 3-payer extension of Prisoner's Dilemma. Our strategies are deterministic and satisfy conditions that are postulated in Axelrod (1984). What is more, ε -good strategies satisfy condition (5) which guarantee that using this strategy our payoff shall not be different than our opponents payoffs for more than ε . This constant ε is totally controlled by the player who choose it. This property is not received by any other author.

Our future aim is to extend the idea presented in Plaskacz (2018) onto the type of games considered in the paper - three players repeated social dilemmas. The idea is as follows. We would like to analyze the repeated three players game by evolutionary games methods. To achieve this goal, we treat the repeated game as a new game in which a player action is a point in the β -core of the original game. Using methods presented in the paper each point from the β -core should determine ε -good strategy. The main difficulty is to obtain the payoff in the case when players choose different points in the β -core. The payoffs in the new game are determined by the payoff in the repeated game.

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APPENDIX

In this Appendix we shall present the proof of Proposition 2.10. We start with the necessary theorem.

Theorem 5.1 If S is a bounded convex subset of H and $V: H \rightarrow \mathbb{R}$ given by (9) is the Lapunov type function for the multivalued map $\varphi: S \rightarrow S$ with the constant $c > 0$, then

$$\exists \gamma > 0, \exists \alpha_0 > 0, \forall \alpha \in [0, \alpha_0], \forall x \in S \setminus \Delta_c, \forall \omega \in \varphi(x) \\ V(\alpha\omega + (1 - \alpha)x) \leq V(x) - \alpha\gamma.$$

Proof: By (10) we choose $\delta \in (0, c)$. Let $M = \sup\{|x|: x \in S\}$. For $x \in S \setminus \Delta_c$ we define a set of indexes $I(x)$ by

$$I(x) = \{j \in \{1, \dots, k\}: V_j(x) \geq V(x) - \delta\}$$

and a subset O_i of S , related to the fixed index i :

$$O_i := \{x \in S \setminus \Delta_c : V_i(x) \geq V(x) - \delta\}. \quad (31)$$

If $x \in O_i$ then $V_i(x) > 0$ and $\langle p_i, \omega \rangle \leq 0$ for all $\omega \in \varphi(x)$. Obviously, $i \in I(x)$ is equivalent to $x \in O_i$ for $x \in S \setminus \Delta_c$.

We fix positive constants: r, γ and α_0 such that

$$r < \frac{\delta}{2}, \quad \gamma < c - \delta, \quad \alpha_0 < \min \left\{ \frac{c - \delta - \gamma}{c - \delta + M}, \frac{r}{2M}, 1 \right\}$$

and take an arbitrary $x \in S \setminus \Delta_c$. The following condition holds true

$$\forall y \in B(x, r) = \{y \in S \setminus \Delta_c : \|x - y\| < r\} \quad \exists i \in I(x) \quad V(y) = V_i(y). \quad (32)$$

Indeed, if $j \notin I(x)$, then $V_j(x) < V(x) - \delta$. Since V_j and V are Lipschitz continuous with the constant $L=1$, we get $V_j(y) < V(y)$. Therefore, there exists $i \in I(x)$ such that $V(y) = V_i(y)$.

If $i \in I(x)$ then $V_i(x) \geq V(x) - \delta \geq c - \delta$ and $\langle p_i, \omega \rangle \leq 0$ for $\omega \in \varphi(x)$. Let $\alpha \in [0, \alpha_0]$ then $x_\alpha := \alpha\omega + (1 - \alpha)x \in B(x, r)$ and $V_i(x_\alpha) \leq V_i(x_\alpha)$. Moreover,

$$V_i(x_\alpha) \geq V_i(x_\alpha) \geq -\alpha_0 \|\omega\| + (1 - \alpha_0)(c - \delta) \geq c - \delta - \alpha_0(c - \delta + M) \geq \gamma$$

and

$$V_i(x_\alpha) \leq (1 - \alpha)V_i(x) \leq V_i(x) - \alpha\gamma.$$

Thus we have obtained that

$$\forall i \in I(x), \quad \forall \alpha \in [0, \alpha_0], \quad \forall \omega \in \varphi(x), \quad V_i(x_\alpha) \leq V_i(x) - \alpha\gamma. \quad (33)$$

The function V has the following property: if $V(a) = V_i(a)$ and $V(b) = V_i(b)$ then $V(\lambda a + (1 - \lambda)b) = V_i(\lambda a + (1 - \lambda)b)$ for $\lambda \in [0, 1]$ so the set

$$\{\alpha \in [0, \alpha_0] : V_i(\alpha\omega + (1 - \alpha)x) = V(\alpha\omega + (1 - \alpha)x)\}$$

is a closed segment. By (32) there exists $s \leq k$ and a partition $0 = \beta_0 < \beta_1 < \dots < \beta_s = \alpha_0$ such that

$$\forall j \in \{0, \dots, s - 1\}, \exists i = i(j) \in I(x), \forall \alpha \in [\beta_j, \beta_{j+1}], \quad V(x_\alpha) = V_i(x_\alpha). \quad (34)$$

Let $\alpha \in [\beta_0, \beta_1]$. In view of (34) there exists $i = i(0) \in I(x)$ such that $V(x_\alpha) = V_i(x_\alpha)$ and by (33):

$$V(x_\alpha) = V_i(x_\alpha) \leq V_i(x) - \alpha\gamma = V(x) - \alpha\gamma.$$

Suppose that

$$V(x_\alpha) \leq V(x) - \alpha\gamma, \quad \forall k = 1, \dots, j-1 \quad \forall \alpha \in [\beta_k, \beta_{k+1}]$$

and take $\alpha \in [\beta_j, \beta_{j+1}]$. By (34) there exists $i = i(j)$ such that $V(x_\alpha) = V_i(x_\alpha)$. Since $\alpha \in [\beta_j, \beta_{j+1}]$, there exists $\xi \in [0,1]$ such that $x_\alpha = \xi\omega + (1 - \xi)x_{\beta_j}$. Therefore,

$$x_\alpha = \xi\omega + (1 - \xi)(\beta_j\omega + (1 - \beta_j)x) = (\xi + (1 - \xi)\beta_j)\omega + (1 - \xi)(1 - \beta_j)x,$$

so

$$\alpha = \xi + (1 - \xi)\beta_j \leq \xi + \beta_j.$$

It is obvious that

$$\begin{aligned} V(x_\alpha) &= V_i(x_\alpha) = V_i(\xi\omega + (1 - \xi)x_{\beta_j}) = \xi \langle \omega, p_i \rangle + (1 - \xi) \langle x_{\beta_j}, p_i \rangle \\ &\leq (1 - \xi)V_i(x_{\beta_j}) = V_i(x_{\beta_j}) - \xi V_i(x_{\beta_j}) \leq V_i(x_{\beta_j}) - \xi\gamma. \end{aligned}$$

Then we get

$$V_i(x_{\beta_j}) - \xi\gamma \leq V(x) - \beta_j\gamma - \xi\gamma = V(x) - \gamma(\beta_j + \xi) \leq V(x) - \alpha\gamma,$$

hence,

$$V(\alpha\omega + (1 - \alpha)x) \leq V(x) - \alpha\gamma. \tag{QED}$$

The proof of Proposition 2.10.

Proof: Fix $(\bar{x}_n)_{n=1}^\infty$ satisfying (12). First, we prove that

$$\forall M, \exists N \geq M, \quad \bar{x}_N \in \Delta_c. \tag{35}$$

Suppose, contrary to our claim, that $\bar{x}_n \notin \Delta_c$ for $n \geq m$. We choose $k \geq m$ such that $\frac{1}{k} < \alpha_0$, where α_0 and γ are given by Theorem 5.1. Thus

$$\begin{aligned} V(\bar{x}_{k+1+1}) &= V\left(\frac{1}{k+1+1}x_{k+1+1} + \frac{k+1}{k+1+1}\bar{x}_{k+1}\right) \leq V(\bar{x}_{k+1}) - \gamma\frac{1}{k+1+1} \leq \\ &\dots \leq V(\bar{x}_k) - \gamma\left(\frac{1}{k+1+1} + \dots + \frac{1}{k+1}\right) \xrightarrow{t \rightarrow \infty} -\infty \end{aligned}$$

which contradicts to the assumption that $V(\bar{x}_n) \geq c$ for $n \geq m$.

Fix $c_1 > c$. By Proposition 2.6, we choose M such that $|\bar{x}_{l+1} - \bar{x}_l| < c_1 - c$ for $l \geq M$. By (35), there exists $N \geq M$ such that $\bar{x}_N \in \Delta_c$. If $\bar{x}_{N+1} \in \Delta_c$ then $V(\bar{x}_{N+1+1}) \leq V(\bar{x}_{N+1}) + |\bar{x}_{N+1+1} - \bar{x}_{N+1}| < c_1$. If $\bar{x}_{N+1} \in \Delta_{c_1} \setminus \Delta_c$ then $V(\bar{x}_{N+1+1}) \leq V(\bar{x}_{N+1}) < c_1$. QED

SILNE I BEZPIECZNE RÓWNOWAGI NASHA W PEWNYCH GRACH POWTARZANYCH 3 GRACZY

Streszczenie

W pracy analizujemy grę nieskończenie powtarzaną 3-graczy będącą rozszerzeniem gry typu Dylemat Więźnia. Rozważamy grę 3-graczy w postaci normalnej z pełną informacją, w której każdy gracz ma dwa działania. Zakładamy, że gra jest symetryczna i powtarzana nieskończenie wiele razy. Strategią gracza w grze powtarzanej jest funkcja zdefiniowana na uwypukleniu zbioru wypłat. Naszym celem jest skonstruowanie mocnej równowagi Nasha w grze powtarzanej, to znaczy profilu strategii, który jest odporny na odstępstwa od strategii równowagi przez koalicję graczy. Skonstruowane strategie równowagi są bezpieczne, to znaczy wypłata gracza, który nie odstępkuje od strategii równowagi jest nie mniejsza od wypłaty odpowiadającej równowadze w grze etapowej, oraz wypłata gracza odstępującego od równowagi może być większa od wypłaty gracza nieodstępującego od strategii równowagi, ale nie więcej niż o pewną stałą dodatnią, która może być wybrana dowolnie małą przez gracza nieodstępującego od równowagi. Nasza konstrukcja jest inspirowana koncepcją dobrych strategii Smale'a opisaną w jego pracy z 1980 roku, gdzie rozważany był powtarzany Dylemat Więźnia. W dowodach wykorzystujemy wyniki o zbliżaniu oraz silnym zbliżaniu.

Słowa kluczowe: gra powtarzana, silna równowaga Nasha, metoda Blackwell'a w problemie zbliżania, metoda funkcji Lapunowa

STRONG AND SAFE NASH EQUILIBRIUM IN SOME REPEATED 3-PLAYER GAMES

Abstract

The paper examines an infinitely repeated 3-player extension of the Prisoner's Dilemma game. We consider a 3-player game in the normal form with incomplete information, in which each player has two actions. We assume that the game is symmetric and repeated infinitely many times. At each stage, players make their choices knowing only the average payoffs from previous stages of all the players. A strategy of a player in the repeated game is a function defined on the convex hull of the set of payoffs. Our aim is to

construct a strong Nash equilibrium in the repeated game, i.e. a strategy profile being resistant to deviations by coalitions. Constructed equilibrium strategies are safe, i.e. the non-deviating player payoff is not smaller than the equilibrium payoff in the stage game, and deviating players' payoffs do not exceed the non-deviating player payoff more than by a positive constant which can be arbitrary small and chosen by the non-deviating player. Our construction is inspired by Smale's good strategies described in Smale's paper (1980), where the repeated Prisoner's Dilemma was considered. In proofs we use arguments based on approachability and strong approachability type results.

Keywords: repeated game, strong Nash equilibrium, Blackwell's approachability, Lapunov function method