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The observational equivalence of natural and unnatural rates of interest

Abstract. *The results of the study presented in this paper demonstrate that a structural model of the natural interest rate, which is consistent with the standard assumptions of the natural rate theory, admits an interpretable, observationally equivalent representation in which a redefined, 'unnatural' equilibrium rate is different from the natural rate in the original model. The alternative representation was obtained by an invertible transformation implemented in the minimal state-space form of the natural-rate model. The identification theory for state-space models is used in the paper to prove the observational equivalence of these two representations. In the alternative representation, the equilibrium interest rate fails to meet the assumption of the natural rate theory, because it depends on past demand shocks. The alternative model, being observationally equivalent, has different implications for the conduct of monetary policy. The problem of observational equivalence arises in relation to natural-rate models because of the inherent unobservability of the natural interest rate; a potential solution to this problem could be the augmentation* of *the information set which is used to identify and estimate the natural rate.*

Keywords: natural rate of interest, state-space model, observational equivalence

JEL Classification: C32, C51, E43

1. INTRODUCTION

The aim of this paper is to demonstrate, using a structural model of the natural (equilibrium) interest rate, that it is possible to find its interpretable, observationally equivalent alternative. The alternative model allows a different interpretation of the equilibrium interest rate and has different implications for the conduct of the monetary policy.

In the 1976 paper, Thomas J. Sargent demonstrated that reduced-form models would not permit to determine the difference between the natural rate theory and its alternatives: 'there are always alternative ways of writing the reduced form, one being observationally equivalent with the other, so that each is equally valid in the estimation period.' (Sargent, 1976, p. 631). Therefore, the rational expectations econometrics, using cross-equation parameter restrictions, has been developed in order to solve the problem of observational equivalence. However, parameter restrictions leave unresolved what Alan J. Preston (1978) called the model identification problem, referring to the fact that there are many models with identified parameters that provide the same fit to the data.

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The natural (equilibrium) rate of interest can be defined as a real rate of interest consistent with real output equalling its potential level in the absence of transitory shocks to demand (Williams, 2003). The potential output is the level of output consistent with the dynamic general equilibrium in the absence of nominal rigidities. Structural models of the natural rate are based on two assumptions, namely, the natural rate of interest is independent of the output gap (difference between actual and potential levels of real output), and a positive output gap cannot be sustained without accelerating inflation.

The model-dependent natural rate of interest is commonly used in the analysis of the monetary policy (see Laubach and Williams, 2003 and 2016; Holston et al., 2017; Fries et al., 2018). The monetary policy stance is defined by the real rate gap (the difference between the measured real rate of interest and the natural rate): a positive real rate gap means a contractionary policy stance, while a negative real rate gap means an expansionary stance. A contractionary or expansionary policy stance is achieved by pursuing policies which vary the real rate of interest with respect to the natural rate. In natural-rate models, monetary policy is neutral with respect to the potential output and the natural rate of interest, thus its scope is limited to the variations of the output gap.

The identification and estimation of the natural rate is usually carried out within a state-space representation of the corresponding structural model where the natural rate is modeled as an unobservable state variable. Two state-space structures are defined to be observationally equivalent if they imply the same probability distribution (likelihood function) for observable variables, and a structure is said to be identifiable if there is no other observationally equivalent structure (Rothenberg, 1971).

A method of identification of state-space structures, developed by Wall (1987), employs a blend of control theory and econometrics. Given that the likelihood function of a linear dynamic system is completely determined by the first two moments, two state-space representations are observationally equivalent if they give rise to the same first two moments of observable variables. Using this property of linear dynamic systems and the concept of minimal representation (a type of representation that includes no redundant state variables), Wall (1987) defined a class of observationally equivalent state-space structures and gave an operational criterion of observational equivalence.

The identification and estimation of state variables require the specification of initial states. For a state-space representation of a stationary multivariate process, initial states can be specified as functions of parameters describing that state-space representation. For a non-stationary process, on the other hand, the state-space representation should be augmented by a model for initial states (see Hamilton, 1994, or Durbin and Koopman, 2012).

A minimal representation, which includes no redundant state variables, guarantees that for given parameter matrices, a change of initial states will imply a change of the likelihood value. For a non-minimal representation, the same value of the likelihood function can be obtained for different initial states and the same parameter matrices. Hence, for a non-minimal representation of a non-stationary process, different realizations of state variables (e.g. the natural rate of interest) can be obtained given the same parameter matrices and the same likelihood value. In other words, a non-minimal state-space representation of a non-stationary process admits an observationally equivalent representation which has the same parameter matrices and different values of state variables.

The natural-rate model considered in this paper is a modification of the widely-used model developed by Laubach and Williams (2003, 2016). The original Laubach-Williams model does not admit an irreducible state-space representation: the model specification used in that model requires the inclusion of redundant state variables (see Appendix). The model considered in this paper is consistent with the assumptions of the natural rate theory and admits an interpretable irreducible state-space representation. This representation is used to obtain an observationally equivalent model, where a redefined, 'unnatural' rate of interest depends on past output gaps, which is called hysteresis, and which involves the dependence of the equilibrium rate on the path the economy experiences towards the equilibrium.

The hysteresis hypothesis explains the fact that the estimates of natural interest rates in advanced industrial economies have been invariably low in the aftermath of the financial crisis of 2007–2008 (see Laubach and Williams, 2016; Holston et al., 2017; Fries et al., 2018). The low estimates of natural interest rates are often explained by persistent deviations from long-run trends ('headwinds', as coined by Yellen, 2015). In natural-rate models, these 'headwinds' come as components of natural interest rates that are exogenous with respect to the output gap. However, the association of recessions with low estimates of natural rates is consistent with the feedback from the output gap to natural rates transferred by "headwinds".

The model with hysteresis has different implications for the conduct of monetary policy: given the fact that the output gap depends on the stance of monetary policy, the equilibrium rate of interest, which depends on past output gaps, can also be affected by monetary policy. And because prolonged recession is likely to cause a downward shift in the equilibrium rate of interest, a more aggressive policy intervention is sometimes necessary in order to close the real rate gap.

The paper consists of four sections. Section 1 introduces the subject of the study, Section 2 provides a brief review of literature on the subject, Section 3 describes observationally equivalent irreducible state-space structures (in general terms), Section 4 demonstrates the observational equivalence of a natural-rate model and a model with hysteresis and Section 5 presents the conclusions of the study.

2. LITERATURE REVIEW

The concept of the natural interest rate, devised by Knut Wicksell (1898), has become popular in empirical research following the publication of Laubach and Williams (2003), in which a small semi-structural model was used to measure the natural rate of interest in the United States. Some modifications of this model were estimated for the Euro Area and other economies (Mésonnier and Renne, 2007; Garnier and Wilhelmsen, 2009; Holston et al., 2017). There are also modifications of the natural-rate model for the open-economy framework (Fries et al., 2018; Wynne and Zhang, 2018a), as well as the attempts to estimate the world natural rate of interest (Wynne and Zhang, 2018b; Kiley, 2019). Although various alternative approaches to the estimation of the natural interest rate have been proposed (Fiorentini et al., 2018; Grossman et al., 2019), the Laubach-Williams model and its modifications have become the most popular empirical tool for measuring the natural rate of interest, frequently cited by policy-makers (Yellen, 2015; Constancio, 2016).

Along with the increasing number of articles utilizing either the Laubach-Williams model or its modifications, there have also been a growing number of papers criticizing this approach. For example, the estimates of the natural rate of interest were found uncertain (Hamilton et al., 2016; Taylor and Wieland, 2016; Beyer and Wieland, 2017), as well as dependent on a priori assumptions concerning the structural relations between unobservable variables (Lewis and Vazquez-Grande, 2017).

Fiorentini et al. (2018) argue that the natural interest rate in the Laubach and Williams (2003) model is unobservable under certain conditions. The authors analyse a state-space representation of a simplified Laubach-Williams model, and demonstrate that state variables, including the determinants of the natural interest rate, are unobservable in two cases – either when the IS curve or the Phillips curve are flat. The unobservability implies that the natural rate is not uniquely identified by the model and the data. Fiorentini et al. (2018) propose a local-level model as an alternative to the Laubach-Williams model.

It can also be demonstrated that the original Laubach-Williams model is not consistent with the observability requirement either (see Appendix). Although a model that is inconsistent with the observability requirement can be transformed into a model that is consistent with that requirement, such transformation would result in a loss of the original economic interpretation. The modification of the Laubach-Williams model that is presented in this paper both fulfils the observability condition and retains the original economic interpretation. For such a model, there is a well-defined class of observationally equivalent models.

3. OBSERVATIONALLY-EQUIVALENT STATE-SPACE STRUCTURES

The state-space representation presented in this paper is

Transition Equation: $\xi_t = F\xi_{t-1} + Gx_t + Qv_t,$ (1)

Measurement Equation: $y_t = H\xi_t,$ (2)

where ξ_t is a $p \times 1$ vector of state variables, y_t is an $n \times 1$ vector of the observed explained variables, x_t is a $k \times 1$ vector of the observed exogenous variables, and v_t is a $q \times 1$ vector of structural shocks; F, G, Q and H are system matrices of dimensions $p \times p$, $p \times k$, $p \times q$ and $n \times p$, correspondingly.

The state-space representation (1)–(2) differs from the state-space representation used in Laubach and Williams (2003), where some dynamic relations between variables were included in the measurement equation. The representation (1)–(2) encompasses all the dynamic relations in the transition equation. Nevertheless, both the original Laubach-Williams model (see Appendix) and the modification considered in this paper admit the representation (1)–(2). This representation facilitates the analysis of observational equivalence without changing structural relations between variables. Because all shocks both in the original Laubach-Williams model and the modification considered in this paper determine the model dynamics, all these shocks appear in the transition equation (1). Although there are no measurement errors in the equation (2), the methodology, described below, would also apply if there were such errors.

The first two moments of explained variables y_t, $\mu_t = E\{y_t\}$ and $\Gamma(t,s) = E\{(y_t - E\{y_t\})(y_s - E\{y_s\})'\}$, are given by

$$\mu(t) = HF'\bar{\xi}_0 + H\sum_{s=1}^{t} F^{t-s}Gx_s,$$

$$\Gamma(t,s) = \begin{cases} HF^{t-s}P_sH' & \text{if } t > s \\ HP_tH' & \text{if } t = s, \\ HP_t(F^{s-t})'H' & \text{if } t < s \end{cases}$$

where $P_t = FP_{t-1}F' + QQ'$ is the covariance matrix of state variables; and $\bar{\xi}_0$ and P_0 are the initial conditions. If the eigenvalues of F are all inside the unit circle, then the process $\{\xi_t\}$ is stationary and the initial conditions are determined by the unconditional moments of this process: $\bar{\xi}_0 = E\{\xi_0\}$ and $P_0 = E\{(\xi_0 - E\{\xi_0\})(\xi_0 - E\{\xi_0\})'\}$. However, if some eigenvalues of F are on the unit circle, then the process $\{\xi_t\}$ is non-stationary and $\bar{\xi}_0$ can represent a guess as to the value of ξ_0 based on prior information, while P_0 measures the uncertainty associated with the guess (Hamilton, 1994).

Following Wall (1987), we say that the two state-space structures $S^{(i)} = \{F^{(i)}, G^{(i)}, Q^{(i)}, H^{(i)}\}$ $(i = 1,2)$ are observationally-equivalent if they produce the same first two moments of y_t. (This definition applies to minimal as well as non-minimal representations).

The state-space representation (1)–(2) is minimal if the dimension of the state-space cannot be reduced without a loss of information about responses of the explained variables y_t to the structural shocks v_t. If the representation (1)–(2) is not minimal, state variables and impulse responses are not uniquely identified and the model cannot be used for policy analysis.

A formal definition of a minimal structure uses the impulse response function $\Phi(h) = \sum_{j=0}^{h} HF^j Q$ and its weighting matrices $W(j) = HF^j Q$. A state-space structure is minimal if for any sequence $j = 0,1, \dots, h$ such that $h \geq (p-1)$, the matrix

$$\begin{bmatrix} HQ \\ HFQ \\ \vdots \\ HF^h Q \end{bmatrix}$$

allows a full-rank decomposition $A(h)B(h)$, where $A(h)$ is an $(h \times n) \times p$ matrix of a full column rank and $B(h)$ is a $p \times q$ matrix of a full row rank (see, e.g., Youla, 1966).

The minimality test is based on checking the rank conditions for the weighting matrices of the structural shocks (Youla, 1966, Lemma 6):

Observability Condition: $$rank \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{p-1} \end{bmatrix} = p, \qquad (3)$$

Controllability Condition: $$rank[Q \; FQ \; \dots \; F^{p-1}Q] = p, \qquad (4)$$

where p is the dimension of the state vector ξ_t.

The observability condition (3) implies that the state-space model (1)–(2) includes no state variables that cannot be inferred from observable variables. The controllability condition (4) implies that the state-space model includes no state variables independent of structural shocks. The observability and controllability are the necessary and sufficient conditions for the minimality of the system (1)–(2). These conditions guarantee a unique identification of state variables and enable the implementation of impulse-response analysis in the model.

The structures $S^{(1)} = \{F^{(1)}, G^{(1)}, Q^{(1)}, H^{(1)}\}$ and $S^{(2)} = \{F^{(2)}, G^{(2)}, Q^{(2)}, H^{(2)}\}$, associated with a minimal state-space representation, are observationally equivalent if and only if there is a non-singular $p \times p$ matrix T such that

$$F^{(2)} = TF^{(1)}T^{-1}, H^{(2)} = H^{(1)}T^{-1},\ G^{(2)} = TG^{(1)}, Q^{(2)} = TQ^{(1)}, \tag{5}$$

where T is a matrix of coordinate transformation: $\xi_t^{(2)} = T\xi_t^{(1)}$. The argument for the equivalence conditions (5) is analogous to the proof of Proposition 1 in Wall (1987). The initial conditions are transformed according to the rule:

$$\bar{\xi}_0^{(2)} = T\bar{\xi}_0^{(1)} \text{ and } P_0^{(2)} = TP_0^{(1)}T^{-1}.$$

The non-singular transformation matrix T is uniquely determined for any pair of minimal representations. A minimal state-space representation is uniquely identified if the only admissible transformation is the identity: $T \equiv I_p$.

The set of minimal representations forms a well-defined class of observational equivalence: there are no observationally-equivalent minimal representations of the same model that have identical parameter matrices and different initial states. Specifying a model, which admits an interpretable minimal representation, gives an operational criterion of observational equivalence.

For non-minimal representations, there is no well-defined class of observational equivalence: non-minimal representations, which have identical parameter matrices and different initial conditions, can be observationally equivalent. It means that there are observationally equivalent representations which have the same parameter matrices but different realizations of state variables, induced by different initial conditions. For a non-stationary (integrated) process, which retains the memory of initial conditions, a unique identification of state variables cannot be obtained in a non-minimal representation. However, observationally equivalent structures can be constructed case-by-case using analytical form of distributional moments.

4. NATURAL-RATE MODEL

Consider a semi-structural econometric model of the natural interest rate which is a modification of the Laubach-Williams model admitting an interpretable minimal representation:

Measured Output: $y_t = y_t^* + \tilde{y}_t,$ (6)

Potential Output: $y_t^* = y_{t-1}^* + g_{t-1} + \sigma_{y^*}\varepsilon_{y^*t},$ (7)

IS Equation: $\tilde{y}_t = a_1\tilde{y}_{t-1} + a_2\tilde{y}_{t-2} + a_r(r_{t-1} - r_{t-1}^*) + \sigma_{\tilde{y}}\varepsilon_{\tilde{y}t},$ (8)

Phillips Curve: $\Delta\pi_t = b_1\Delta\pi_{t-1} + b_2\Delta\pi_{t-2} + b_3\Delta\pi_{t-3} + b_{\tilde{y}}\tilde{y}_{t-1} + \sigma_\pi\varepsilon_{\pi t},$ (9)

Potential Growth: $g_t = g_{t-1} + \sigma_g\varepsilon_{gt},$ (10)

'Headwinds': $$z_t = \rho_z z_{t-1} + \sigma_z \varepsilon_{zt}, \qquad (11)$$

Natural Rate: $$r_t^* = c g_t + z_t, \qquad (12)$$

where y_t is the logarithm of the measured output in period t; y_t^* and $\tilde{y}_t$ are the (unobservable) potential output and output gap; g_t is the growth rate of the potential output; π_t is the inflation rate; r_t is the measured real rate of interest; r_t^* is the unobservable natural rate of interest; $(r_t - r_t^*)$ is the real rate gap; z_t is the non-growth component of the natural interest rate ('headwinds'); and ε_{y^*t}, $\varepsilon_{\tilde{y}t}$, $\varepsilon_{\pi t}$, ε_{gt}, ε_{zt} are structural shocks which are independent over time and across variables and have the standard normal distribution. Note that there are no measurement errors in the model – all the shocks enter dynamic equations and drive the dynamics of the system.

The parameters of the model are assumed to satisfy the following restrictions: $a_r < 0$, $b_y > 0$ and $|\rho_z| \leq 1$. The parameters σ_g and σ_z cannot be identified in the model (6)–(12) because of the 'pileup problem', discussed by Stock (1994). Laubach and Williams (2003) estimate these parameters using the median-unbiased estimator described in Stock and Watson (1998). The application of the median-unbiased estimator requires an additional assumption: $\rho_z = 1$. For the estimation of the full system (6)–(12), the parameters σ_g and σ_z are set to be equal to their median-unbiased estimators obtained at preliminary stages.

The potential output y_t^* is modeled as an I(2) variable and the growth rate g_t is assumed to be a random walk. If the persistence parameter ρ_z in equation (11) equals one, the process $\{z_t\}$ is a random walk and, as results from equation (12), the natural rate of interest r_t^* and the growth rate of the potential output g_t can diverge. For values of ρ_z smaller than one, the process $\{z_t\}$ is stationary and the natural rate of interest r_t^* is cointegrated with the growth rate of the potential output g_t.

All the equations in the model (6)–(12), except for the IS equation (8), are equivalent to the equations in the original Laubach-Williams model. The IS equation (8) includes only one lag of the real rate gap $(r_t - r_t^*)$, and this modification guarantees the observability of state variables. The original Laubach-Williams model, including two lags of the real rate gap in the IS equation, fails the observability condition (see Appendix). The failure of the observability condition implies that the natural rate of interest is not uniquely identified in the model. A minimal form of the Laubach-Williams model is derived in the Appendix. However, it is only possible to obtain it by such a transformation of state variables that the Laubach-Williams model loses its original interpretation.

The model (6)–(12) retains the economic interpretation of the Laubach-Williams model and has a minimal state-space representation. It is consistent with all the standard assumptions of the natural rate theory. The potential output y_t^* and its growth rate g_t are exogenous with respect to the output gap $\tilde{y}_t$. The non-growth component of the natural interest rate z_t is also modeled as exogenous with respect to the output gap $\tilde{y}_t$, and, consequently, the natural rate of interest $r_t^* = cg_t + z_t$ is exogenous with respect to the output gap $\tilde{y}_t$. The 'accelerationist' Phillips curve (9) implies that a positive output gap accelerates inflation. Nevertheless, the model can be rewritten in an observationally equivalent form, which admits feedback from the output gap to the non-growth component of the natural interest rate:

Measured Output: $y_t = y_t^* + \tilde{y}_t,$ (6*)

Potential Output: $y_t^* = y_{t-1}^* + g_{t-1} + \sigma_{y^*}\varepsilon_{y^*t},$ (7*)

IS Equation: $\tilde{y}_t = a_1\tilde{y}_{t-1} + \tilde{a}_2\tilde{y}_{t-2} + a_r(r_{t-1} - \tilde{r}_{t-1}^*) + \sigma_{\tilde{y}}\varepsilon_{\tilde{y}t},$ (8*)

Phillips Curve: $\Delta\pi_t = b_1\Delta\pi_{t-1} + b_2\Delta\pi_{t-2} + b_3\Delta\pi_{t-3} + b_{\tilde{y}}\tilde{y}_{t-1} + \sigma_\pi\varepsilon_{\pi t},$ (9*)

Potential Growth: $g_t = g_{t-1} + \sigma_g\varepsilon_{gt},$ (10*)

'Headwinds': $\tilde{z}_t = \rho_z\tilde{z}_{t-1} + \alpha(\tilde{y}_{t-1} - \rho_z\tilde{y}_{t-2})\sigma_z\varepsilon_{zt},$ (11*)

Natural Rate: $\tilde{r}_t^* = cg_t + \tilde{z}_t,$ (12*)

where the non-growth component of the natural rate is redefined as $\tilde{z}_t = z_t + \alpha\tilde{y}_{t-1}$ and α is a positive constant. This model can be obtained by an invertible coordinate transformation in a state-space representation of model (6)–(12). The coordinate transformation is a linear transformation of a state vector that generates an observationally equivalent state-space representation, where some or all components of the transformed state vector are different from the components of the original state vector. It also changes structural relations between the components of state vector. The coordinate transformation is achieved by a pre-multiplication of the state vector by an invertible matrix, which determines the changes.

The transformed model retains the same structural form of the IS and the Phillips curve equations. Only one parameter of the IS equation changes: $\tilde{a}_2 = a_2 + \alpha a_r < \mathrm{a}_2$. The modified IS equation (8*) satisfies all the restrictions imposed in original IS equation (8), although it includes a different equilibrium rate: $\tilde{r}_t^* = r_t^* + \alpha\tilde{y}_{t-1}$. The transformed model includes a hysteresis effect: the past demand shocks affect the current value of the equilibrium rate $\tilde{r}_t^*$.

The model (6)–(12) can be written in the state-space form (1)–(2), where

$$\mathbf{y}_t = \begin{bmatrix} y_t \\ \Delta\pi_t \end{bmatrix}, x_t = [r_{t-1}], H = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\xi_t = \begin{bmatrix} y_t^* \\ \tilde{y}_t \\ \tilde{y}_{t-1} \\ g_t \\ z_t \\ \Delta\pi_t \\ \Delta\pi_{t-1} \\ \Delta\pi_{t-2} \end{bmatrix}, F = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & a_1 & a_2 & -a_r c & -a_r & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho_z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & b_y & 0 & 0 & 0 & b_1 & b_2 & b_3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, G = \begin{bmatrix} 0 \\ \frac{a_r}{2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$Q = \begin{bmatrix} \sigma_{y^*} & 0 & 0 & 0 & 0 \\ 0 & \sigma_{\tilde{y}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_g & 0 & 0 \\ 0 & 0 & 0 & \sigma_z & 0 \\ 0 & 0 & 0 & 0 & \sigma_\pi \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } v_t = \begin{bmatrix} \varepsilon_{y^*t} \\ \varepsilon_{\tilde{y}t} \\ \varepsilon_{gt} \\ \varepsilon_{zt} \\ \varepsilon_{\pi t} \end{bmatrix}.$$

This state-space representation is irreducible (minimal), i.e. it includes no redundant state variables. The irreducibility is implied by the satisfied observability and controllability conditions:

$$rank \begin{bmatrix} H \\ HF \\ HF^2 \\ HF^2 \\ HF^3 \\ HF^4 \\ HF^5 \\ HF^6 \\ HF^7 \end{bmatrix} = 8 \text{ and } rank[Q \;\; FQ \;\; F^2Q \;\; F^3Q \;\; F^4Q \;\; F^5Q \;\; F^6Q \;\; F^7Q] = 8,$$

where the rank is equal to the number of state variables. (The Python routine that implements these rank tests is available from the author upon request).

The vector of state variables ξ_t includes non-stationary (integrated) variables and requires an initialization model (see Hamilton, 1994 or Durbin and Koopman, 2012). The initialization model is not discussed in this paper. For any pair of observationally equivalent minimal state-space representations, there is a unique invertible transformation of initial conditions (which would not be the case for non-minimal representations).

The coordinate transformation matrix, which generates the model (6*)–(12*) from the model (6)–(12), is

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The system matrices of the transformed model $\tilde{H} = HT^{-1}$, $\tilde{G} = TG$ and $\tilde{Q} = TQ$ are identical to the corresponding matrices of the original model, except for the transition matrix $\tilde{F} = TFT^{-1}$:

$$\tilde{F} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & a_1 & a_2 + \alpha a_r & -a_r c & -a_r & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \alpha & -\alpha\rho_z & 0 & \rho_z & 0 & 0 & 0 \\ 0 & b_{\tilde{y}} & 0 & 0 & 0 & b_1 & b_2 & b_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The observational equivalence follows from the minimality of the original state-space representation and the invertibility of transformation matrix T.

For a non-minimal model, such as the original Laubach-Williams model, there are multiple realizations of state variables (including the natural rate r_t^*) that are induced by different initial conditions and are observationally equivalent for a given structure (see Appendix).

5. CONCLUSIONS

This paper demonstrates that it is possible to transform a structural model of the natural (equilibrium) interest rate in such a way as to obtain an interpretable observationally equivalent model in which the redefined 'unnatural' interest rate is different, because it depends on past output gaps.

Specifying a model that admits a minimal state-space representation allows a class of observationally equivalent models to be defined. For a well-defined class of observationally equivalent models it is easier to identify interpretable alternatives and envision model modifications which would exclude such alternatives.

The cause of the model non-uniqueness is the inherent unobservability of the natural (equilibrium) rate, which is determined by other unobservable variables and does not directly enter into an equation for an observable variable. It allows redefining the equilibrium rate by reshuffling other unobservable variables without losing information about observable variables. A potential solution to this problem is to augment the information set which is used for the identification and estimation of the natural rate, for example, the dynamics of the non-growth component of the natural rate (z_t) can be explained by some observable variables. Yellen (2015) lists some of the 'headwinds' that determine the non-growth component of the natural interest rate. Augmenting a natural-rate model with exogenous observable variables that explain the dynamics of z_t would restrict the class of observationally equivalent models.

The unobservability problem in the Laubach-Williams model can potentially be solved by imposing economically-motivated restrictions on the initial values of state variables, or by re-specifying the dynamics of natural-rate components. For example, specifying g_t and z_t as second-order autoregressive processes may solve the problem of unobservability. However, it would require either the estimation or the calibration of additional parameters.

The issue of the model non-uniqueness becomes important when there is an observationally equivalent model which admits a meaningful alternative interpretation. In the example presented in this paper, the hysteresis effect in the transformed model can explain the persistent shift in the level of the equilibrium interest rate caused by a demand-driven recession. This interpretation is consistent with persistently low real interest rates in many advanced industrial economies in the aftermath of the financial crisis of 2007–2008. The model, in which there is a feedback from the output gap to the equilibrium interest rate, has particular implications for the monetary policy, namely, it calls for a more active monetary policy response to contractionary demand shocks.

APPENDIX. MINIMAL REPRESENTATION OF THE LAUBACH-WILLIAMS MODEL

Consider the original Laubach-Williams model:

Measured Output: $y_t = y_t^* + \tilde{y}_t,$ (13)

Potential Output: $y_t^* = y_{t-1}^* + g_{t-1} + \sigma_{y^*}\varepsilon_{y^*t},$ (14)

IS Equation: $\tilde{y}_t = a_1\tilde{y}_{t-1} + a_2\tilde{y}_{t-2} + \frac{a_r}{2}\sum_{j=1}^{2}\left(r_{t-j} - r_{t-j}^*\right) + \sigma_{\tilde{y}}\varepsilon_{\tilde{y}t},$ (15)

Phillips Curve: $\pi_t = b_\pi\pi_{t-1} + (1 - b_\pi)\bar{\pi}_{t-2,4} + b_{\tilde{y}}\tilde{y}_{t-1} + \sigma_\pi\varepsilon_{\pi t},$ (16)

Potential Growth: $g_t = g_{t-1} + \sigma_g\varepsilon_{gt},$ (17)

'Headwinds': $z_t = z_{t-1} + \sigma_z\varepsilon_{zt},$ (18)

Natural Rate: $r_t^* = cg_t + z_t,$ (19)

where y_t is the logarithm of the measured output in the period *t*; y_t^* and $\tilde{y}_t$ are the (unobservable) potential output and output gap; g_t is the growth rate of the potential output; π_t is the inflation rate; r_t is the measured real rate of interest; $\bar{\pi}_{t-2,4}$ is the average of inflation over periods $t-2$, $t-3$ and $t-4$; $\bar{\pi}_{t-2,4} = (\pi_{t-2} + \pi_{t-3} + \pi_{t-4})/3$; r_t^* is the unobservable natural rate of interest; $(r_t - r_t^*)$ is the real rate gap; z_t is the non-growth component of the natural interest rate ('headwinds'); and ε_{y^*t}, $\varepsilon_{\tilde{y}t}$, $\varepsilon_{\pi t}$, ε_{gt}, ε_{zt} are structural shocks which are independent over time and across variables and have the standard normal distribution.

Using the definition of $\bar{\pi}_{t-2,4}$ and the restriction on inflation lags in the equation (16), it can be rewritten as:

Phillips Curve: $$\Delta\pi_t = b_1\Delta\pi_{t-1} + b_2\Delta\pi_{t-2} + b_3\Delta\pi_{t-3} + b_{\tilde{y}}\tilde{y}_{t-1} + \sigma_\pi\varepsilon_{\pi t}, \quad (16^*)$$

where $b_1 = (b_\pi - 1)$, $b_2 = \frac{2}{3}(b_\pi - 1)$ and $b_3 = \frac{1}{3}(b_\pi - 1)$. The equations (16) and (16*) are equivalent representations of the Phillips curve.

Since all disturbances (ε_{y^*t}, $\varepsilon_{\tilde{y}t}$, $\varepsilon_{\pi t}$, ε_{gt} and ε_{zt}) in the model (13)–(19) are structural shocks entering dynamic equations, it can be written in the state-space representation (1)–(2) where

$$\xi_t = \begin{bmatrix} y_t^* \\ \tilde{y}_t \\ \tilde{y}_{t-1} \\ g_t \\ g_{t-1} \\ z_t \\ z_{t-1} \\ \Delta\pi_t \\ \Delta\pi_{t-1} \\ \Delta\pi_{t-2} \end{bmatrix}, F = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_1 & a_2 & -\frac{a_r c}{2} & -\frac{a_r c}{2} & -\frac{a_r}{2} & -\frac{a_r}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho_z & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & b_y & 0 & 0 & 0 & 0 & 0 & b_1 & b_2 & b_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$G = \begin{bmatrix} 0 & 0 \\ \frac{a_r}{2} & \frac{a_r}{2} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\mathbf{y}_t = \begin{bmatrix} y_t \\ \Delta\pi_t \end{bmatrix}, x_t = \begin{bmatrix} r_{t-1} \\ r_{t-2} \end{bmatrix}, H = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$Q = \begin{bmatrix} \sigma_{y^*} & 0 & 0 & 0 & 0 \\ 0 & \sigma_{\tilde{y}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_g & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_z & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_\pi \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } v_t = \begin{bmatrix} \varepsilon_{y^*t} \\ \varepsilon_{\tilde{y}t} \\ \varepsilon_{gt} \\ \varepsilon_{zt} \\ \varepsilon_{\pi t} \end{bmatrix}.$$

This state-space representation is non-minimal: it includes two redundant (unobservable) states. The redundancy follows from the failure of the observability rank condition (The Python routine that implements the rank test is available from the author upon request):

$$rank \begin{bmatrix} H \\ HF \\ HF^2 \\ HF^2 \\ HF^3 \\ HF^4 \\ HF^5 \\ HF^6 \\ HF^7 \\ HF^8 \\ HF^9 \end{bmatrix} = 8 < 10.$$

The source of the redundancy is the imbalance in the dynamics of state variables: although variables g_t and z_t are defined as first-order autoregressive processes (random walks), two lags of each variable are included in the IS equation:

$$\tilde{y}_t = a_1\tilde{y}_{t-1} + a_2\tilde{y}_{t-2} + \frac{a_r}{2}\sum_{j=1}^{2}\left(r_{t-j} - r^*_{t-j}\right) + \sigma_{\tilde{y}}\varepsilon_{\tilde{y}t} =$$

$$a_1\tilde{y}_{t-1} + a_2\tilde{y}_{t-2} + \frac{a_r}{2}\sum_{j=1}^{2}\left(r_{t-j} - cg_{t-j} - z_{t-j}\right) + \sigma_{\tilde{y}}\varepsilon_{\tilde{y}t}.$$

A minimal representation can be obtained by applying a decomposition which is analogous to the decomposition used in Youla (1966, Corollary 2). The coordinate transformation of the state vector, which can be implemented to obtain the decomposition, is given by the invertible matrix below:

$$T=\begin{bmatrix}T^{(0)}\\T^{(1)}\end{bmatrix}=\left[\begin{array}{cccccccccc}1&0&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0\\0&0&1&0&\frac{a_rc}{2a_2}&0&\frac{a_r}{2a_2}&0&0&0\\0&0&0&1&0&0&0&0&0&0\\0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&0&1\\\hline 0&0&0&1&-1&0&0&0&0&0\\0&0&0&0&0&\frac{1}{\rho_z}&-1&0&0&0\end{array}\right].$$

The sub-matrix $T^{(0)}$ composed of the first eight rows of the matrix T generates a minimal state vector $\xi_t^{(0)}=T^{(0)}\xi_t$. The sub-matrix $T^{(1)}$ composed of the last two rows of the matrix T generates a vector of redundant states $\xi_t^{(1)}=T^{(1)}\xi_t$ which affect neither the minimal state vector $\xi_t^{(0)}$ nor the vector of explained variables y_t. Because of the redundancy, the transformation matrix T is not unique.

The structure of the minimal state-space system is

$$\xi_t^{(0)}=\begin{bmatrix}y_t^*\\\tilde{y}_t\\w_t\\g_t\\z_t\\\Delta\pi_t\\\Delta\pi_{t-1}\\\Delta\pi_{t-2}\end{bmatrix},\ F^{(0)}=\begin{bmatrix}1&0&0&1&0&0&0&0\\0&a_1&a_2&-\frac{a_rc}{2}&-\frac{a_r}{2}&0&0&0\\0&1&0&\frac{a_rc}{2a_2}&\frac{a_r}{2a_2}&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&\rho_z&0&0&0\\0&b_y&0&0&0&b_1&b_2&b_3\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\end{bmatrix},\ G^{(0)}=\begin{bmatrix}0&0\\\frac{a_r}{2}&\frac{a_r}{2}\\0&0\\0&0\\0&0\\0&0\\0&0\\0&0\\0&0\end{bmatrix},$$

$$\mathbf{y}_t=\begin{bmatrix}y_t\\\Delta\pi_t\end{bmatrix},\ x_t=\begin{bmatrix}r_{t-1}\\r_{t-2}\end{bmatrix},\ H=\begin{bmatrix}1&1&0&0&0&0&0&0\\0&0&0&0&0&1&0&0\end{bmatrix},$$

where neither the new state variable $w_t=\tilde{y}_{t-1}+\frac{a_r}{2a_2}(cg_{t-1}+z_{t-1})$ nor the transformed equation describing the dynamics of the output gap have a meaningful economic interpretation:

$$\tilde{y}_t=a_1\tilde{y}_{t-1}+a_2w_{t-1}+\frac{a_r}{2}(r_{t-1}-r_{t-1}^*)+\frac{a_r}{2}\,r_{t-2}+\sigma_{\tilde{y}}\varepsilon_{\tilde{y}t}. \qquad (20)$$

The equation (20) cannot be interpreted as an IS equation. If we try to restore the original model (13)–(19) from its minimal form by inverting the transformation T, then any choice of initial values y_{-1}, g_{-1} and z_{-1} such that $w_0 = \tilde{y}_{-1} + \frac{a_r}{2a_2}(cg_{-1} + z_{-1})$ does not change will generate an observationally equivalent model with a different natural rate of interest.

Hendry (1995, p. 36) defines both the uniqueness and the interpretability as necessary conditions for the model identification. In the original Laubach-Williams model, state variables are consistent with the assumed interpretation, but they are not unique (multiple realizations of the natural interest rate are possible in the same model). If the Laubach-Williams model is reduced to a minimal representation, state variables are uniquely identified, but the model loses the assumed interpretation. The problem of identification should be reconsidered at an earlier stage, when the model specification is selected. The model specification (6)–(12) guarantees both the uniqueness of state variables and their interpretability.

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