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LETTER FROM THE EDITOR

Dear Readers,

I have great pleasure in presenting the first 2020 issue of *Przegląd Statystyczny. Statistical Review*. I am proud to announce that in order to maintain the journal's high quality and to extend its range and impact, the Editors and the Publisher have agreed to introduce several changes to both the journal and the editorial process. From now on, *Przegląd Statystyczny. Statistical Review* will publish articles exclusively in English. The submitted manuscripts are screened for plagiarism by means of the Crossref Similarity Check programme. Papers accepted for publication are subject to thorough proofreading, which is free of charge and usually significantly improves the readability of the paper. Research papers appear online prior to being published in printed and electronic formats. All publications issued since 2009 are stored in an online archive which is easily accessible through the journal's web page. Papers published since 2016 have all been provided with digital object identifiers. We have also extended the abstracting and indexing of the journal by including it in several new databases (Central and Eastern European Online Library – CEEOL, ICI Journals Master List, ICI World of Journals, and the Norwegian Register for Scientific Journals, Series and Publishers – the Nordic List). Additionally, we launched a new web page of the journal in order to increase its recognisability.

In 2019, 30 papers were submitted for publication in our journal, out of which 16 were accepted and subsequently published in four consecutive issues. The number of accepted papers was the same as in the year before. So far in 2020, we have accepted five papers for publication.

We would like to kindly invite members of academia and practitioners to submit papers to *Przegląd Statystyczny. Statistical Review*. We can guarantee that original research on theoretical and empirical subjects relating to statistics, econometrics, mathematical economics, operational research, decision sciences, and data analysis will be published after going through a prompt, yet thorough editorial process. We would also be pleased to receive high quality papers written by PhD candidates, where advances in research and topical issues from all fields of economics, finance, and management are discussed.

On behalf of the Board of Editors,
Paweł Miłobędzki
Editor-in-Chief

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Evaluation of the financial condition of companies after the announcement of arrangement bankruptcy: application of the classical and Bayesian logistic regression

Barbara Pawełek,[a] Jadwiga Kostrzewska,[b] Maciej Kostrzewski,[c] Krzysztof Gałuszka[d]

Abstract. The aim of this paper is to present the results of an assessment of the financial condition of companies from the construction industry after the announcement of arrangement bankruptcy, in comparison to the condition of healthy companies. The logistic regression model estimated by means of the maximum likelihood method and the Bayesian approach were used. The first achievement of our study is the assessment of the financial condition of companies from the construction industry after the announcement of bankruptcy. The second achievement is the application of an approach combining the classical and Bayesian logistic regression models to assess the financial condition of companies in the years following the declaration of bankruptcy, and the presentation of the benefits of such a combination. The analysis described in the paper, carried out in most part by means of the ML logistic regression model, was supplemented with information yielded by the application of the Bayesian approach. In particular, the analysis of the shape of the posterior distribution of the repeat bankruptcy probability makes it possible, in some cases, to observe that the financial condition of a company is not clear, despite clear assessments made on the basis of the point estimations.

Keywords: company, arrangement bankruptcy, financial condition, Maximum Likelihood Method, Bayesian approach, logistic regression

JEL: C11, C25, G33

1. Introduction

Company bankruptcy is an important issue for economic sciences. The establishment of new companies and the termination of business activity by some existing companies are natural phenomena in a free-market economy. Nevertheless, the situation where a company announces bankruptcy, and, consequently, discontinues its business activity has been the object of research carried out by scientists, economic practitioners, and financial institutions. This increased interest might be explained by serious social and economic consequences of bankruptcies, therefore there is a need for developing methods of bankruptcy prediction.

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The paper by Sun et al. (2014) contains a review of about 140 publications dated 1966–2014 on bankruptcy prediction problems, with respect to e.g. the definition of disadvantageous financial condition of a company and company bankruptcy. The authors emphasised the diversity of definitions of the above-mentioned phenomena. They also demonstrated that theoretical arguments define three levels of a disadvantageous financial condition, while empirical studies are usually limited to the analysis of just two conditions: a healthy company and a bankrupt company.

Bankruptcy is awarded to a company by a court. In the period from 1 October 2003 to 31 December 2015, a court could decide on two types of bankruptcy with regard to insolvent companies: bankruptcy open to arrangements and liquidation bankruptcy.[1] The former enables a company to continue its business activity on condition that it complies with procedures enabling it to pay off as much of its debt as possible.

In the meantime, the company bankruptcy law in Poland was amended to the effect that now, each bankruptcy is announced as business liquidation. However, the new regulation,[2] in force since 1 January 2016, foresees, instead of arrangement bankruptcy, four new procedures leading to an arrangement between the debtor and the creditors, thus giving debtors more opportunities to solve their problems. Additionally, the legislator introduced the possibility of an arrangement in the case of insolvency, which is applied in order to give a company a chance to survive in a situation where opening or continuing a restructuring procedure is impossible. The aim of the new restructuring law is to increase the efficiency of procedures leading to an arrangement between the debtor and the creditors.

Predicting bankruptcy is a frequent topic in literature describing the application of multidimensional statistical analysis to the business sector. However, only few papers focus on the statistical assessment of the financial condition of companies in the years following the declaration of bankruptcy. Examining the process of overcoming insolvency issues might become a valuable source of information that is useful in the assessment of a success probability in the case of the accomplishment of restructuring proposals by subsequent bankrupts.

The aim of this paper is to present the results of the assessment of the financial condition of companies from the construction industry after the announcement of arrangement bankruptcy in comparison to the condition of healthy companies. The logistic regression model estimated by means of the maximum likelihood method and the Bayesian approach were used for this purpose. The research hypothesis claims that the application of the Bayesian approach to the logistic regression model

[1] Ustawa z dnia 28 lutego 2003 r. – *Prawo upadłościowe*, Dz.U. 2003 nr 60 poz. 535.
[2] Ustawa z dnia 15 maja 2015 r. – *Prawo restrukturyzacyjne*, Dz.U. 2015 poz. 978.

guarantees the enrichment of conclusions on a company's post-arrangement-bankruptcy financial condition drawn on the basis of the logistic regression model estimated by means of the maximum likelihood method. The first achievement of our study is the assessment of the financial condition of companies from the construction industry in relation to which arrangement bankruptcy has been announced. This means they try to deal with their solvency issues following the bankruptcy arrangements affirmed by a court. The second achievement involves applying a combination of the classical and Bayesian logistic regression models in the process of assessing the financial condition of bankrupt companies in the years following the announcement of bankruptcy, and presenting the benefits of such a combination. The results of our pilot research for the years 2005 and 2009 were previously presented in Kostrzewska et al. (2016).

The remaining part of the paper is organised in the following way: Section 2 presents the state of research on bankrupt companies, Section 3 describes the process of data preparation, Section 4 details the methodology of the empirical examination, Section 5 shows the results of calculations, their interpretation and graphical presentation, Section 6 discusses the received results in the light of knowledge on further histories of the analysed bankrupt companies, Section 7 summarises the reflections on the topics covered in the paper. The list of literature referred to in the work is included at the end of the paper.

2. State of research

Scientists have been researching efficient bankruptcy prediction methods since as early as the 20th century. The most popular bankruptcy prediction methods include for instance: the linear discriminant function (e.g. Altman, 1968; García et al., 2019; Lee and Choi, 2013), the logit model (e.g. Ohlson, 1980; Li and Wang, 2014; Tseng and Hu, 2010), the classification tree (e.g. Frydman et al., 1985; Abellán and Castellano, 2017; Tsai et al., 2014), the neural network (e.g. Odom and Sharda, 1990; López et al., 2015; Tkáč and Verner, 2016), the support vector machine (e.g. Liang et al., 2015; Sun et al., 2017; Zhou et al., 2015), the hazard model (e.g. Beaver et al., 2005; Beaver et al., 2012; Shumway, 2001), and the ensemble method (e.g. Ekinci and Erdal, 2017; Pawełek, 2019; Zhou and Lai, 2017). The above-mentioned types of analyses are based on sets of financial data obtained from healthy and bankrupt companies dated usually a year or two before the bankruptcy occurred. The Bayesian approach in forecasting bankruptcy was also applied and described in literature (e.g. Sarkar and Sriram, 2001; Sun and Shenoy, 2007; Trabelsi et al., 2015), whereas the Bayesian logistic regression has not yet been used in the presented context, as far as we know.

Scientific literature on company bankruptcies also discusses issues relating to insolvent companies' activity ensuing their bankruptcy. For instance, Eberhart et al. (1999) examined reactions of the capital market to bankruptcies of listed companies. In their analysis, the authors applied, for example cumulative abnormal returns to analyse changes in share prices of bankrupt companies in a period of 200 days following the bankruptcy announcement.

Another important issue is the assessment of the risk of repeat bankruptcy of companies (e.g. Platt and Platt, 2002; Altman and Branch, 2015). Analyses are conducted on the basis of financial data of bankrupt companies. In order to predict the threat of repeat bankruptcy, Platt and Platt (2002) used the logit model. The research set consisted of 51 bankrupt companies, 9 out of which were subject to bankruptcy procedures for the second time. The model included three explanatory variables: annual abnormal return, net sales one year after re-emerging from bankruptcy, and the number of months spent in the first bankruptcy process. The classification accuracy of the model was high, i.e. 90.2% in total, 88.9% in the group of repeat bankrupts and 90.5% in the group of bankrupts continuing their business activity. On the basis of the estimated model, it was demonstrated that a decrease in annual abnormal return, a decrease in net sales one year after re-emerging from bankruptcy, and an increase in the number of months spent in the first bankruptcy process might increase the probability of repeat bankruptcy *ceteris paribus*. Platt and Platt (2002) presented the similarities and differences between their research and the results presented in the work of Hotchkiss (1995).

Altman and Branch (2015) measured the usefulness of the Altman Z-Score formula intended for predicting the success or failure of a company's post--bankruptcy business activity. The linear discriminant function was determined on the basis of the following financial indicators: Current Assets – Current Liabilities / Total Assets, Retained Earnings / Total Assets, EBIT / Total Assets and Book Value of Equity / Total Liabilities. In their research, the authors analysed two groups of companies – one group that consisted of companies which experienced problems with continuing their business activity only once, and the other comprising businesses which had become bankrupt twice. The research set consisted of 148 bankrupt companies, 61 of which had difficulty in continuing their business activity. According to the authors, both courts and persons responsible for restructuring of companies should employ statistical methods to predict whether a company is prone to repeat bankruptcy as a form of supplementing the traditional analysis. Such methods may facilitate the assessment of the restructuring plan and the monitoring of the post-bankruptcy condition of a company in order to adjust the plan accordingly. A return to court means that the restructuring failed in terms of the concept, moreover generating social and economic costs. The authors emphasised the importance of an early-warning system which would reduce the probability of the occurence of repeat bankruptcy, often preceded by a long and costly restructuring process.

An alternative approach to the above-mentioned concepts is the statistical evaluation of the financial condition of companies in the years following the announcement of arrangement bankruptcy compared against the condition of healthy companies (e.g. Kostrzewska et al., 2016; Pawełek et al., 2017). Within this approach, analyses are based on financial data of both healthy companies and companies which have announced bankruptcy. Thus, this solution makes it possible to research methods of solving bankruptcy problems by companies when data sets on bankrupt companies are not comprehensive enough. This approach might prove helpful in the selection of an appropriate recovery programme for companies experiencing solvency problems. However, caution is recommended when interpreting the results. It is because provisions of the tax law, policies of financial institutions, etc. are likely to influence the assessment of the economic condition of a company and its further existence on the market. Additionally, differences in accounting regulations, particularly in terms of the interpretation of international accounting standards, make it difficult – or even impossible – to compare data between countries. As a consequence, the examination of the financial condition of companies after arrangement bankruptcy must be conducted for each country separately.

Moreover, no or limited comparability may occur within one country, as some companies are subject to audits of their financial statements, whereas others are not obliged to do so. In Poland there are companies with bad financial indicators that do not go bankrupt, but are doing well due to tax provisions allowing the use of financial losses when a company merges with another company with losses. On the other hand, financial institutions are distrustful of companies with bad financial indicators and seek to receive their funds quickly by means of arrangement bankruptcy. Therefore, a company in a poorer financial condition may survive while a company in a better financial condition may go bankrupt if the latter has higher debts in financial institutions. What was said above shows the difficulties in assessing the financial condition of companies after arrangement bankruptcy is announced.

3. Data processing

3.1. Database

The data used for the analysis have been collected from the Emerging Markets Information Service (EMIS).[3] The research covered 369 construction companies in

[3] http://www.emis.com.

Poland, including five companies undergoing arrangement bankruptcy (B_1–B_5). The court awards were issued between November 2003 and August 2004. The research used all 14 financial ratios available in the EMIS database which have been grouped as follows: liquidity ratios (R_{01}–R_{03}), liability ratios (R_{04}–R_{06}), profitability ratios (R_{07}–R_{10}), and productivity ratios (R_{11}–R_{14}) (Table 1).

The financial data come from the years 2005–2009. The same legal bankruptcy and recovery provisions were in force in Poland in the period when companies under research were announcing arrangement bankruptcy and in the years for which companies' financial statements were available.

Table 1. Financial ratios

Symbol	Description	Symbol	Description
R_{01}	Current liquidity ratio	R_{08}	Net profitability
R_{02}	Quick liquidity ratio	R_{09}	ROE
R_{03}	Cash Ratio	R_{10}	ROA
R_{04}	Total Debts to Assets	R_{11}	Accounts Receivable Turnover
R_{05}	Debt to Equity	R_{12}	Fixed Asset Turnover
R_{06}	Long-term debt to Equity	R_{13}	Total Asset Turnover
R_{07}	Gross profitability	R_{14}	Operation cost to sales revenues

Note. For interested readers, descriptive statistics of financial ratios for both analysed groups are available upon request.
Source: based on information from the Emerging Markets Information Service.

The empirical research was conducted on the basis of an unbalanced set. Sets of this kind, more often than balanced sets, are characterised by low accuracy of classification of bankrupt companies in the considered bankruptcy prediction methods. The above-mentioned phenomenon may result from a small proportion of bankrupts in the examined sets as well as, for instance, from the existence of untypical objects among healthy companies (e.g. Pawełek et al., 2015). An untypical company is understood by the authors as an object with outlying values of financial ratios. The studies conducted so far showed that the removal of these outliers (in this case the untypical objects among healthy companies) from a data set before performing estimations raises the classification accuracy of the logistic regression model (e.g. Pawełek et al., 2015).

Thus, the preparation of the data for analysis includes also the detection of outliers. An outlier is an element that seems to be considerably different from other elements of a set in which it is included (e.g. Grubbs, 1969; Barnett and Lewis, 1994; Hodge and Austin, 2004; John, 1995). An outlier is often an element at an incomparably larger distance from other values in the set than the distances between other elements of the set (Johnson and Wichern, 1992). Papers on the examination of the financial condition of companies, especially those focusing on bankruptcy prediction, might discuss the presence of outliers. The proposed solutions of this problem include ignoring it (e.g. Spicka, 2013), replacement or deletion of outliers (e.g. De Andrés et al., 2011; Shumway, 2001; Wu et al., 2010), or the application of robust methods (e.g. Hauser and Booth, 2011).

Untypical healthy companies, defined as above, may be characterised by both a very good and a bad financial condition, which, with regard to numerous indicators, is similar to the condition of companies in announced bankruptcy. The detection and deletion of untypical healthy companies in/from a set of objects also has a substantive justification. Economic practice shows that companies in a bad financial condition (i.e. whose financial ratios are of disadvantageous values) might not be able to fulfil the prerequisites necessary to initiate bankruptcy proceedings or the obligation to file a bankruptcy petition. If no such petition is filed by creditors either, such companies exist on the market and affect the condition of the entire sector.

3.2. Detection of outliers

Due to the above, outliers were deleted from the data sets prior to the estimation of the parameters of logistic regression models. To detect outliers, two one-dimensional methods were used, based on the quantile analysis or the Tukey criterion (Tukey, 1977) and a multidimensional method based on the projection depth function (Zuo, 2003). Each method was applied on the basis of all 14 discussed financial ratios, or – in the case of the one-dimensional methods – in combination with the discriminatory power analysis (e.g. Yu et al., 2014), which uses financial ratios with stronger discriminatory power than others. Altogether, five methods were applied to detect untypical healthy companies. When interpreting the results of the analysis, it should be remembered that the financial ratios of typical healthy companies reflected the financial condition of the construction industry in a given year, which depended, for instance on the economic situation in Poland.

The procedure based on the Tukey criterion required calculations of the first quartile ($Q_{0.25}$) and the third quartile ($Q_{0.75}$) for every financial ratio in the group of healthy companies, and then the calculation of the interquartile range ($Q = Q_{0.75} - Q_{0.25}$). Values outside of the range of $\langle Q_{0.25} - 1.5Q, Q_{0.75} + 1.5Q \rangle$ were considered outliers. A healthy company was considered untypical if at least one financial ratio value was an outlier.

The detection of an outlier by means of the quantile analysis was performed in the following way: the values of quantile $Q_{0.10}$ (in the case of strong left-sided asymmetry of the financial ratio distribution), quantile $Q_{0.90}$ (in the case of strong right-sided asymmetry) or quantiles $Q_{0.05}$ and $Q_{0.95}$ (if there was no strong asymmetry) were defined for every financial ratio in the group of healthy companies. If a company had values of a given ratio lower than $Q_{0.10}$, higher than $Q_{0.90}$ or falling outside the range defined by quantiles $Q_{0.05}$ and $Q_{0.95}$, depending on the type of asymmetry observed, it was considered an outlier. As in the case of the Tukey criterion, a healthy company was considered untypical if at least one value of the financial ratios was an outlier.

The projection depth function (Zuo, 2003) was applied to detect outliers in a multi-dimensional space. The concept of data depth is related to the nonparametric robust multi-dimensional statistical analysis developed within the scope of exploratory data analysis. This enables the determination of a linear order of multi-dimensional observations with the use of a multi-dimensional median defined as a multi-dimensional centre of a set of observations (Zuo and Serfling, 2000). There are numerous depth functions available which assign a positive number to each observation derived from a certain distribution, determining its distance from the centre (e.g. Kosiorowski, 2008). We assumed arbitrarily that outlier healthy companies are the 10% of items located at the greatest distance from a multi--dimensional centre. Here, it has to be remembered that the projection depth function used to detect outliers is a method which indicates items at a large distance from the centre of the data set, regardless of the direction of these items' location (i.e. the group of the above-mentioned outlier companies may include both companies in a very good financial condition and companies experiencing serious financial problems).

3.3. Discriminatory power of financial ratios

In order to establish which ratios are best in signalling a company's deteriorating financial condition, their discriminatory power was determined. The first approach was based on ratio distribution quantiles (e.g. Yu et al. 2014). The number of bankrupts belonging to the 10-percent range of extreme values for healthy companies (in the right or left tail of the distribution) was assumed as the criterion. When values adopted by the bankrupts were present in both tails of the distribution for healthy companies, then the percentage of bankrupts was checked in the two-sided 10-percent range defined by quantiles $Q_{0.05}$ and $Q_{0.95}$. The higher the value of this criterion, the higher the discriminatory power of a given ratio. If in one of the areas defined by quantiles for healthy companies the ratio values were defined for at least two out of five bankrupt companies (which was a very mild criterion), then the ratio was considered as having a discriminatory power. The sets of financial ratios with a discriminatory power differed throughout the analysed years (Table 2).

In particular, at the beginning of the examined period, that is in 2005, as many as 12 ratios had a discriminatory power (calculated according to the adopted criterion), while in the three subsequent years – only 9 to 4 ratios. It means that the values of the financial ratios of construction companies shortly after the announcement of arrangement bankruptcy were in the tails of the distributions of ratios for healthy companies. The longer the time passed since the bankruptcy announcement, the fewer ratios had the discriminatory power and, consequently, the fewer bankrupt companies were characterised by extreme values of the financial ratios. The number of ratios with a discriminatory power increased from 4 variables in 2008 to 7 variables in 2009, which may have resulted from the global financial crisis. In Poland, its repercussions included the deterioration of conditions for business activity within the construction industry in 2009.

Table 2. Financial ratios with higher discriminatory power than other discussed ratios – the analysis based on the asymmetry and quantiles of distribution

Year	Financial ratios	m
2005	R_{01} R_{02} R_{04}* R_{05} R_{07} R_{08} R_{09}** R_{10} R_{11} R_{12} R_{13} R_{14}**	12
2006	R_{04}* R_{05} R_{06} R_{07}** R_{08}** R_{10}** R_{11} R_{12} R_{14}*	9
2007	R_{05} R_{06}** R_{09} R_{14}*	4
2008	R_{03}** R_{05} R_{06}** R_{09}	4
2009	R_{03} R_{07} R_{08} R_{09}**R_{10} R_{11} R_{14} *	7

Note. m – number of ratios, * – discriminatory power in the right tail, ** – discriminatory power in both tails, other – discriminatory power in the left tail.
Source: authors' calculation.

The financial ratios with a higher discriminatory power than the other examined ratios (calculated according to the adopted criterion – Table 2) in individual years usually include representatives of three out of four groups. The only exception is the year 2005, i.e. shortly after the announcement of arrangement bankruptcy, when selected ratios from all four examined groups had a discriminatory power. Liquidity ratios did not occur in sets specified for the years 2006 and 2007. Liability ratios were missing for 2009, while productivity ratios were not specified for 2008. Selected profitability ratios were included in all sets.

The second approach was based on the Tukey criterion with the hinge distance factor equal to 1.5. The number of bankrupts with ratio values belonging to the specified ranges of extreme values for healthy companies was the adopted criterion. The higher the value of this criterion, the bigger the discriminatory power of a given ratio. The financial ratios with discriminatory power were those for which the values of the ratios were recorded for at least two out of five bankrupt companies in areas defined by the Tukey criterion with the hinge distance factor equal to 1.5 for healthy companies. At the beginning of the examined period, i.e. in 2005, only six financial ratios had a discriminatory power (calculated in accordance with the adopted criterion). In the following years, the number of ratios with a discriminatory power decreased from three variables in 2006 and 2007 to one variable in 2008 and 2009. The approach based on the Tukey criterion with the hinge distance factor equal to 1.5 was more restrictive than the approach based on the quantiles of ratio distribution.

Table 3. Financial ratios with higher discriminatory power than other discussed ratios – the analysis based on the Tukey criterion

Year	Financial ratios	m
2005	R_{05} R_{07}** R_{08}** R_{09}** R_{10} R_{14}**	6
2006	R_{05} R_{06} R_{08}**	3
2007	R_{05} R_{066}** R_{09}**	3
2008	R_{06}**	1
2009	R_{14}*	1

Note. See Table 2.
Source: authors' calculation.

There were no liquidity ratios with a higher discriminatory power than other analysed ratios (Table 3) in the examined years in the case of the Tukey criterion. Liability ratios did not occur in the set specified for 2009. Profitability ratios were missing in 2008 and 2009, while the productivity ratios were not specified for the years 2006–2008.

3.4. Sets of companies used in logistic regression models

To determine outliers for individual years, the one-dimensional analysis method based on the Tukey criterion and the quantile analysis, as well as the multidimensional method based on the projection depth function were used. The methods were indicated with their first letters (T meant the analysis based on the Tukey criterion, Q – the quantile analysis, D – the projection depth function), which were followed by a two-digit symbol of a year (05, 06, 07, 08, and 09) and the number of financial ratios used to detect outliers (m = 4, 7, 9, 12, 14 in the analysis based on quantiles; m = 1, 3, 6, 14 in the analysis based on the Tukey criterion). If m does not equal 14, it means that the discriminatory power of financial ratios was taken into account in a given method.

In further analyses, sets established by means of the above-mentioned methods were examined. The following variants were adopted:

- variant I – sets including all companies in a given year (complete database),
- variant II – sets including all bankrupts and healthy companies other than untypical items for a given method in a given year.

The number of elements in the sets in variant II for individual years are presented in Table 4.

Table 4. Numbers of elements in the sets in variant II depending on the method applied and the number of financial ratios (m)

Method	Variant II				
	2005	2006	2007	2008	2009
$T.yy.14$	188 ($m = 14$)	205 ($m = 14$)	197 ($m = 14$)	176 ($m = 14$)	188 ($m = 14$)
$T.yy.m$	292 ($m = 6$)	320 ($m = 3$)	309 ($m = 3$)	321 ($m = 1$)	353 ($m = 1$)
$Q.yy.14$	174 ($m = 14$)	190 ($m = 14$)	167 ($m = 14$)	169 ($m = 14$)	179 ($m = 14$)
$Q.yy.m$	213 ($m = 12$)	247 ($m = 9$)	269 ($m = 4$)	272 ($m = 4$)	268 ($m = 7$)
$D.yy.14$	333 ($m = 14$)	333 ($m = 14$)	333 ($m = 14$)	333 ($m = 14$)	333 ($m = 14$)

Note. $T.yy.14$ ($T.yy.m$) – the method based on the Tukey criterion applied for data from the year $20yy$ and concerning 14 (m) financial ratios, $Q.yy.14$ ($Q.yy.m$) – the method based on quantiles applied for data from the year $20yy$ and concerning 14 (m) financial ratios, $D.yy.14$ – the method based on the projection depth function applied for data from year $20yy$ and concerning 14 (m) financial ratios.
Source: authors' calculation.

In the case of the one-dimensional analysis based on the Tukey criterion or the quantile analysis including/not including information on the discriminatory power of financial ratios for different years, various numbers of elements in variant II (Table 4) were received, as, due to the values of different ratios, the outliers do not necessarily need to be the same. In the case of the multi-dimensional analysis based on the projection depth function, sets containing the same numbers of elements in variant II (333 companies) were obtained for different years. However, in the case of both the one-dimensional methods (depending on whether the discriminatory power of ratios was taken into account) and the multi-dimensional analysis, the same items do not necessarily have to be outliers. Thus, differences may occur in logistic regression models assessed using knowledge based on outliers that have been obtained through various methods.

4. Research methodology

To assess the financial condition of construction companies in Poland, the following logistic regression model was applied:

$$P(y_i = \text{bankrupt}\,\mathbf{x_i}) = \frac{\exp(\mathbf{x_i}\boldsymbol{\beta} + \varepsilon_i)}{1 + \exp(\mathbf{x_i}\boldsymbol{\beta} + \varepsilon_i)} \tag{1}$$

where:
y_i – the dependent variable for the i-th object,
$\mathbf{x_i}$ – the vector of explanatory variables for the i-th object,
$\boldsymbol{\beta}$ – the vector of parameters,
ε_i – the error term.

On this basis, the companies were classified into two groups: a group of entities in a financial condition typical for healthy companies (i.e. non-bankrupt – 'NB' Group) and a group of entities in a financial condition typical for companies with announced arrangement bankruptcy (i.e. bankrupt – 'B' Group). Classification into the 'B' Group in the case of bankrupt companies meant that they were prone to repeat bankruptcy, while companies classified into the 'NB' Group were in a financial condition similar to that of healthy companies.

To estimate the parameters of the logistic regression model, the maximum likelihood method and the Bayesian approach were used. Explanatory variables were selected by means of the backward stepwise regression method. As a result, the maximum likelihood logistic regression model (ML logistic regression model) and the Bayesian logistic regression model were built on the same sets of explanatory variables. These approaches were applied jointly in order to supplement the results obtained in the classical approach with the information obtained through the Bayesian approach.

As far as the ML logistic regression model is concerned, the authors made point and interval estimations of the probability of a company falling into a group of items in a financial condition typical for companies with announced arrangement bankruptcy, i.e. a group of items threatened with first or repeat bankruptcy (hereinafter referred to as 'bankruptcy probability'). For each company, point estimates of bankruptcy probability were assessed using respective values of financial ratios within the ML logistic regression model. Interval estimates of bankruptcy probability for every company were obtained in compliance with the methodology described in the paper by Neter et al. (1989). The logit transformation was used for the calculations.

A company was classified into the group of entities exposed to the risk of bankruptcy if the point estimate of bankruptcy probability calculated on the basis of the ML logistic regression model was higher than 0.5. In the case of 95% confidence intervals for bankruptcy probability, a company was considered prone to bankruptcy if the lower bound of the confidence interval was above 0.5. If the upper bound of the confidence interval was below 0.5, a company was considered to be in a financial condition typical for healthy companies in a given year. However, if the lower bound of the confidence interval was below 0.5, while the upper was above 0.5, the financial condition of a given entity was considered ambiguous.

The Bayesian inference is based on the posterior distribution, which combines prior knowledge and information provided by data within a mathematical model. The approach can be applied regardless of the size of the data set. Moreover, it takes into account the uncertainty regarding unknown parameters. Therefore, the Bayesian approach provides the distribution of an unknown quantity, unlike a single point or interval estimate, which is the case for the maximum likelihood method.

In the Bayesian approach, in order to express the lack of prior knowledge for the individual parameters of the logistic regression model, relatively uninformative prior independent normal distributions $N(0, 10)$ were assumed. The expected value equal to zero corresponded to the assumption concerning the statistical insignificance of the model parameters. In other words, neither negative nor positive signs of the parameters were priorly preferable, while the standard deviation of 10 made the prior distribution diffuse. The random walk Metropolis-Hastings algorithm was used to sample from the posterior distribution of the parameters (Gamerman and Lopes, 2006). The starting points of the algorithm were assumed at the level of the maximum likelihood estimates. The acceptance rates of the numerical algorithm were high and exceeded 40 percent. The presented results of the Bayesian inference were based on 100,000 MCMC draws, preceded by 50,000 burn-in Markov chain cycles.

It was assumed that if the point estimate of the median of the posterior distribution of the bankruptcy probability was higher than 0.5, such a company was classified into the group of items threatened with first or repeat bankruptcy. The classification on the basis of the expected value of the posterior distribution led to the same conclusions. The examination of the financial condition of companies was also based on a graphical analysis of the posterior distribution of the probability of a company belonging to the 'B' Group, presented by means of histograms.

The classification accuracy of the estimated logistic regression models was assessed by means of the following measures (e.g. Birdsall, 1973; Fawcett, 2006; Krzanowski and Hand, 2009): *sensitivity* – calculated as a percentage of companies in announced arrangement bankruptcy that were classified in the 'B' Group; *specificity* – calculated as a percentage of healthy companies that were classified in the 'NB' Group; *AUC measure* – the area under the ROC curve. Due to a high percentage of healthy companies and a small percentage of bankrupts, the *sensitivity* and *AUC measures* were chosen as the classification accuracy measures, recommended in literature for imbalanced sets (García et al., 2015; Kostrzewska et al., 2016). The measures of the classification accuracy were calculated on the basis of a full data set (variant I), i.e. including the outliers. Therefore, it was possible to maintain the comparability of the calculated measures between individual models built on the basis of various data sets cleaned by means of various methods of outlier detection.

5. Empirical results

Following the removal of outliers from the data set by means of any of the above-mentioned methods, in order to classify the companies according to the financial condition either into the 'NB' Group (i.e. of entities whose financial condition resembles that of a healthy company) or the 'B' Group (i.e. of entities in a financial condition typical for companies in announced arrangement bankruptcy), the authors estimated 25 ML logistic regression models and 25 corresponding Bayesian logistic regression models (5 ML models and 5 Bayesian models for every year in the period 2005–2009). As mentioned before, the reduction of the input set of explanatory variables was performed by means of the backward stepwise regression method. The list of variables left in the logistic regression model estimated for individual years in the period 2005–2009 is presented in Table 5.

Table 5. Financial ratios in the logistic regression models after reduction of the input set of explanatory variables

Model	Explanatory variables in the logistic regression model				
	2005	2006	2007	2008	2009
$MT.yy.14$ $MTB.yy.14$	R_{04}	R_{05} R_{09}	R_{05} R_{06}	R_{05} $R_{06}R_{10}$ R_{14}	R_{06} R_{14}
$MT.yy.m$ $MTB.yy.m$	R_{10} R_{14}	R_{05}	R_{05} R_{06}	R_{06}	R_{14}
$MQ.yy.14$ $MQB.yy.14$	R_{04}	R_{04} R_{05} R_{08}	R_{05} R_{06}	R_{05} $R_{06}R_{10}$ R_{14}	R_{14}
$MQ.yy.m$ $MQB.yy.m$	R_{04}	R_{04} R_{05} R_{08}	R_{05} R_{06}	R_{05} R_{06}	R_{14}
$MD.yy.14$ $MDB.yy.14$	R_{04}	R_{04} R_{05} R_{08}	R_{05} $R_{06}R_{08}$ R_{14}	R_{05} $R_{06}R_{11}$ R_{14}	R_{09}

Note. $MT.yy.14$ ($MTB.yy.14$) – the ML logistic regression model (the Bayesian logistic regression model) built on the basis of data cleaned with a method based on the Tukey criterion dated $20yy$ and referring initially to 14 financial ratios, $MT.yy.m$ ($MTB.yy.m$) – the ML model (the Bayesian model) built on the basis of data cleaned with a method based on the Tukey criterion dated $20yy$ and referring initially to m financial ratios, $MQ.yy.14$ ($MQB.yy.14$) – the ML model (the Bayesian model) built on the basis of data cleaned with a method based on quantiles dated $20yy$ and referring initially to 14 financial ratios, $MQ.yy.m$ ($MQB.yy.m$) – the ML model (the Bayesian model) built on the basis of data cleaned with a method based on quantiles dated $20yy$ and referring initially to m financial ratios, $MD.yy.14$ ($MDB.yy.14$) – the ML model (the Bayesian model) built on the basis of data cleaned with a method based on the projection depth function dated $20yy$ and referring initially to 14 financial ratios.
Source: authors' calculation using the *STATISTICA* software.

The analysis of the set of financial ratios with a statistically significant impact on the probability of belonging to the group of entities in a financial condition typical for companies in announced arrangement bankruptcy led to the conclusion that the role of liability ratios was of particular importance – mainly in the years 2005–2008.

The liquidity ratios did not have a significant impact on the bankruptcy probability in the analysed group of companies and in the reviewed period. Profitability ratios were present every year but for different variants of models. The role of productivity ratios was of particular importance for the assessment of bankruptcy risk in the years 2008 and 2009.

When predicting bankruptcies on the basis of unbalanced sets, it is advisable to use the *sensitivity* measure followed by the AUC measure first, due to a high percentage of healthy companies and a low percentage of bankrupts. The high values of the *specificity* measure may result from a large share of healthy companies in the sample. Tables 6 and 7 present the values of *sensitivity* and AUC measures calculated for the logistic regression model prepared by means of the maximum likelihood method and the Bayesian approach.

Table 6. Sensitivity measure for the ML and the Bayesian logistic regression models estimated on the basis of sets without outliers

Model	Sensitivity				
	2005	2006	2007	2008	2009
$MT.yy.14$	**0.8**	**0.6**	**0.6**	**0.8**	0.4
$MTB.yy.14$	**0.8**	**0.6**	**0.6**	0.4	0.2
$MT.yy.m$	0.2	0.4	**0.6**	0.2	0.4
$MTB.yy.m$	0.4	0.4	**0.6**	0.2	0.2
$MQ.yy.14$	**0.8**	**0.8**	**0.6**	**0.6**	**0.6**
$MQB.yy.14$	**0.8**	**0.8**	**0.6**	0.4	0.2
$MQ.yy.m$	**0.8**	**0.8**	0.4	0.2	**0.6**
$MQB.yy.m$	**0.8**	**0.8**	0.4	0.2	0.4
$MD.yy.14$	**0.8**	**0.6**	0.4	0.2	0.2
$MDB.yy.14$	**0.8**	**0.6**	0.4	0.2	0.2

Note. See Table 5, *specificity* measures higher than 0.5 are shown in bold.
Source: authors' calculation using the *STATISTICA* software.

On the basis of information presented in Tables 6 and 7, it may be stated that the values of the *sensitivity* and AUC measures for the ML logistic regression models and Bayesian logistic regression models are usually similar. The largest differences were observed for the *sensitivity* measures in the years 2008 and 2009, when the assessment of bankruptcy risk calculated by means of the Bayesian model was milder than in the case of the ML model.

Table 7. *AUC* measure for the ML and the Bayesian logistic regression models estimated on the basis of sets without outliers

Model	*AUC*				
	2005	2006	2007	2008	2009
$MT.yy.14$	**0.945**	**0.904**	0.764	**0.903**	**0.857**
$MTB.yy.14$	**0.945**	**0.891**	0.764	**0.879**	**0.852**
$MT.yy.m$	0.706	0.744	0.757	0.607	**0.824**
$MTB.yy.m$	0.726	0.747	0.762	0.629	**0.826**
$MQ.yy.14$	**0.945**	**0.911**	0.759	**0.908**	**0.822**
$MQB.yy.14$	**0.945**	**0.900**	0.764	**0.874**	**0.826**
$MQ.yy.m$	**0.945**	**0.901**	0.770	0.792	**0.833**
$MQB.yy.m$	**0.945**	**0.895**	0.770	0.790	**0.826**
$MD.yy.14$	**0.942**	**0.888**	**0.898**	**0.932**	0.772
$MDB.yy.14$	**0.945**	**0.885**	**0.817**	**0.927**	0.772

Note. See Table 5, *AUC* measures higher than 0.8 are shown in bold.
Source: authors' calculation using the *STATISTICA* software.

Out of the considered logistic regression models (i.e. estimated on the basis of data sets with outliers deleted by means of various methods), the authors selected models with the highest *sensitivity* and *AUC* measures for further analysis. Values of the *specificity* measure were also taken into consideration in the analysis. The examination of these measures led to the conclusion that taking account of the discriminatory power of financial ratios usually did not improve the classification accuracy measured by the *sensitivity* and *AUC* measures. Among the models estimated on sets constructed without the use of the discriminatory power analysis, the highest values of the *sensitivity*, *specificity* and *AUC* measures were usually observed in the case of models built on the basis of the quantile analysis. The advantage of this solution was minor compared to the solution based on the Tukey criterion. Therefore, further analysis was conducted on the basis of the models *MQ.yy*.14 and *MQB.yy*.14, where *yy* = 05, 06, 07, 08, 09, i.e. the ML logistic regression and the Bayesian logistic regression models estimated on the sets cleaned by means of the quantile analysis.

The aim of the study was to assess the financial condition of five bankrupt companies in the years 2005–2009, i.e. shortly after they announced arrangement bankruptcy. Therefore, the classification of healthy companies was ignored in the discussion on the results of the analysis.

Table 8 presents, for each bankrupt company B_1–B_5, the point estimates of the probability of belonging to the 'B' Group of entities (whose financial condition is typical for companies with announced arrangement bankruptcy) and 95% confidence intervals for the repeat bankruptcy probability determined by means of the ML logistic regression model (*MQ.yy*.14).

Table 8. Point and interval (in round brackets) estimates of the probability of belonging to a group of entities in a financial condition typical for companies with announced arrangement bankruptcy, calculated for each bankrupt company B_1–B_5 by means of the ML logistic regression model

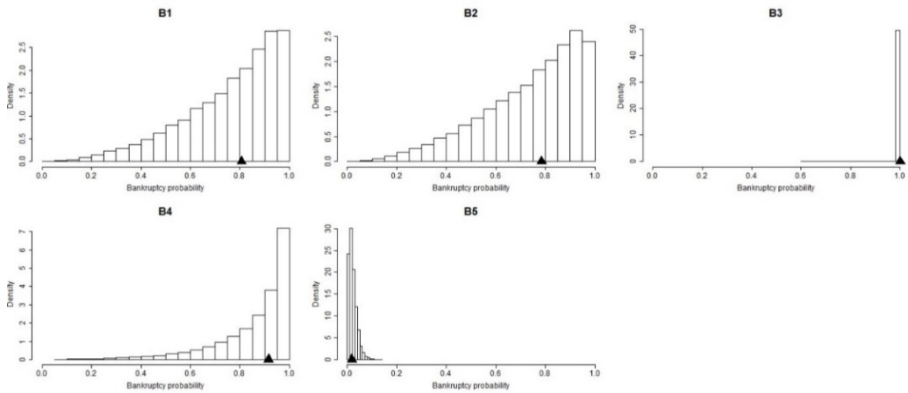
Year	Company with announced arrangement bankruptcy				
	B_1	B_2	B_3	B_4	B_5
2005	0.9408	0.9301	1.0000	0.9823	0.0123
	(0.1401, 0.9994)	(0.1338, 0.9991)	(0.8651, 1.0000)	(0.1894, 0.9999)	(0.0015, 0.0961)
2006	0.9101	1.0000	0.9235	0.9885	0.0045
	(0.0202, 0.9998)	(0.9999, 1.0000)	(0.0271, 0.9998)	(0.0355, 1.0000)	(0.0005, 0.0400)
2007	0.9996	0.6787	0.0192	0.6691	0.0024
	(0.0297, 1.0000)	(0.0440, 0.9898)	(0.0051, 0.0702)	(0.0525, 0.9866)	(0.0003, 0.0225)
2008	0.9959	0.8137	0.0102	0.9706	0.4152
	(0.1329, 1.0000)	(0.0518, 0.9972)	(0.0010, 0.0964)	(0.0005, 1.0000)	(0.0542, 0.8979)
2009	0.9953	0.0064	0.0170	0.6553	0.7272
	(0.6086, 1.0000)	(0.0011, 0.0369)	(0.0043, 0.0651)	(0.1593, 0.9502)	(0.1824, 0.9695)

Source: authors' calculation using the *STATISTICA* software.

Considering the point estimates of the probability of bankrupt companies B_1–B_5 belonging to the 'B' Group of entities, as determined by means of the ML logistic regression model for the years 2005–2009 (Table 8), it may be stated, in accordance with the adopted criterion, that the financial condition typical for healthy companies was observed for company B_2 in 2009, B_3 in 2007–2009 and B_5 in 2005–2008. In general, 95% confidence intervals confirmed these conclusions. The assessment of the financial condition of bankrupt company B_5 in 2008 was the only exception as the interval estimation indicated an unclear situation of this company. Furthermore, the risk of repeat bankruptcy established on the basis of the point estimations of the bankruptcy probability (17 cases – see Table 8) was confirmed by the interval estimations with respect to only three bankrupts, i.e. B_1 in 2009, B_2 in 2006 and B_3 in 2005. In the remaining 14 cases, the interval estimates indicated an unclear financial condition of the analysed bankrupts.

An unclear assessment of the financial condition of bankrupts on the basis of the point and interval estimations of repeat bankruptcy probability led to performing the Bayesian analysis on the 25 analysed cases. Figures 1–5 present histograms of the posterior distribution of the repeat bankruptcy probability for individual bankrupt companies B_1–B_5, with medians yielded by the Bayesian logistic regression model (*MQB.yy*.14).

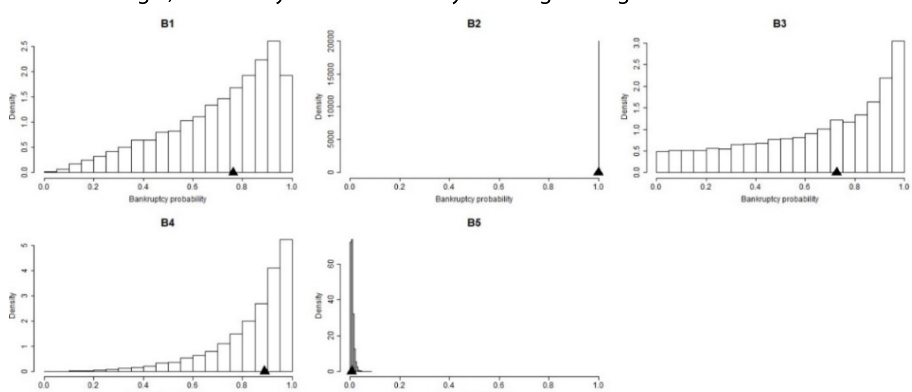
Figure 1. Histograms of the posterior distribution of the probability of bankrupt companies B_1–B_5 belonging to the group of entities in a financial condition typical for companies with announced arrangement bankruptcy, and median values (marked with a black triangle) defined by means of the Bayesian logistic regression model for 2005



Source: authors' calculation using the *MCMCpack* package in R.

The analysis of the medians and the shape of the histograms of the posterior distribution of the probability of bankrupt companies belonging to the group of entities in a financial condition typical for companies with announced arrangement bankruptcy (Figure 1) in 2005 shows the threat of repeat bankruptcy of bankrupts B_1–B_4. The shapes of the histograms obtained for bankrupts B_1 and B_2 indicate that the risk of repeat bankruptcy for these companies is lower than in the case of bankrupt B_4. As far as company B_5 is concerned, the median value and the histogram show that the financial condition of this bankrupt in 2005 was similar to the condition typical for healthy companies.

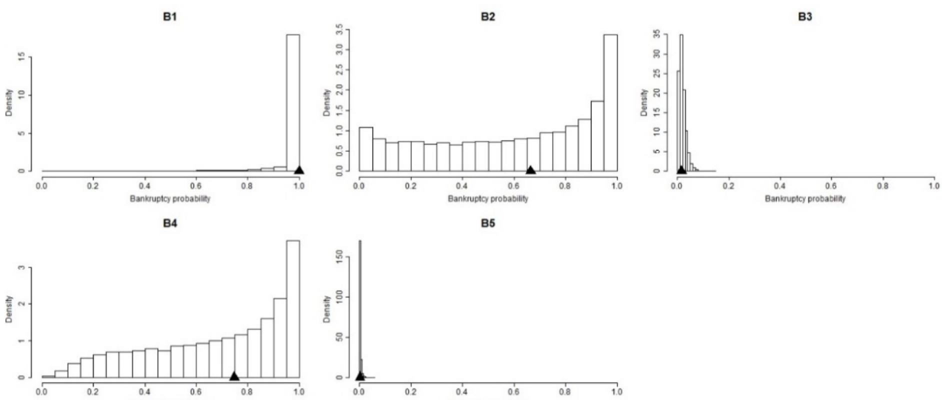
Figure 2. Histograms of the posterior distribution of the probability of bankrupt companies B_1–B_5 belonging to the group of entities in a financial condition typical for companies with announced arrangement bankruptcy, and median values (marked with a black triangle) defined by means of the Bayesian logistic regression model for 2006



Source: authors' calculation using the *MCMCpack* package in R.

The medians and histograms of the posterior distribution of bankruptcy probability calculated for 2006 (Figure 2) show the threat of repeat bankruptcy of bankrupts B_1–B_4. In the case of company B_5, the median indicates the similarity of this bankrupt's financial condition to the condition typical for a healthy company. The histograms of the posterior distribution of the repeat bankruptcy probability for companies B_1, B_2, and B_4 strengthen the conclusions made on the basis of the medians, and bankrupt B_2 seems to be more threatened with repeat bankruptcy than bankrupts B_1 and B_4. It is difficult to defend the conclusion regarding the threat of repeat bankruptcy of bankrupt B_3 obtained on the basis of the medians, as the histogram for this company shows that the assessment of its financial condition is unclear. On the basis of the shape of the histogram obtained for bankrupt B_5, it is possible to uphold the conclusion made on the basis of the medians.

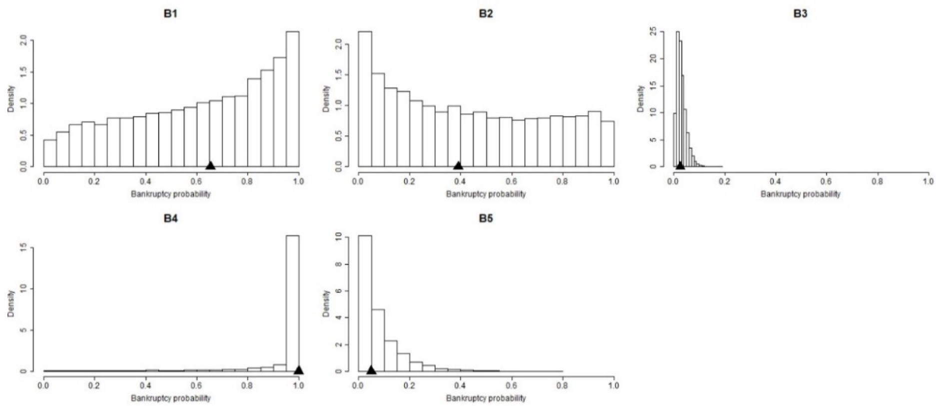
Figure 3. Histograms of the posterior distribution of the probability of bankrupt companies B_1–B_5 belonging to the group of entities in a financial condition typical for companies with announced arrangement bankruptcy, and median values (marked with a black triangle) defined by means of the Bayesian logistic regression model for 2007



Source: authors' calculation using the *MCMCpack* package in R.

The medians of the posterior distribution of bankruptcy probability calculated for 2007 (Figure 3), demonstrate that for bankrupts B_1, B_2, and B_4 there is a threat of repeat bankruptcy. The histogram of the posterior distribution of bankruptcy probability for bankrupt B_1 definitely justifies the conclusion made on the basis of the median. The shapes of the histograms in the case of bankrupts B_2 and B_4 allow only for a relatively unclear assessment of their financial condition. The medians calculated for companies B_3 and B_5 suggest the similarity between their financial conditions and that of a healthy company. The analysis of the histograms reinforces these conclusions.

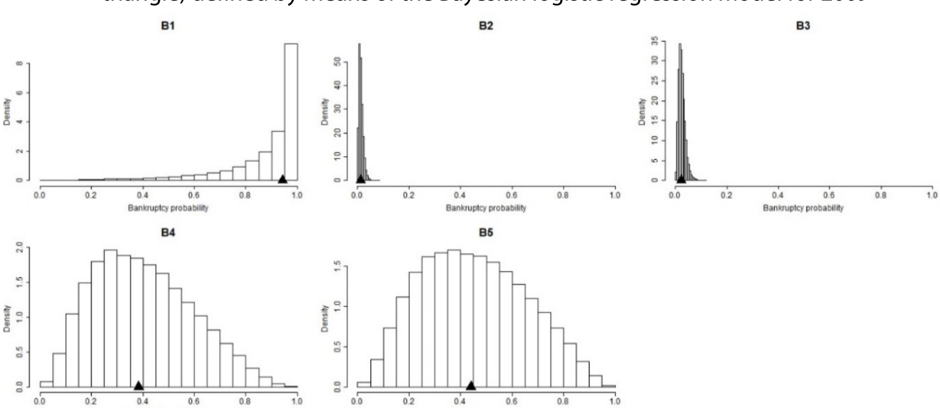
Figure 4. Histograms of the posterior distribution of the probability of bankrupt companies B_1–B_5 belonging to the group of entities in a financial condition typical for companies with announced arrangement bankruptcy, and median values (marked with a black triangle) defined by means of the Bayesian logistic regression model for 2008



Source: authors' calculation using the *MCMCpack* package in R.

The analysis of the medians of the posterior distribution of the bankruptcy probability calculated for 2008 (Figure 4) leads to the conclusion that bankrupts B_1 and B_4 are threatened with repeat bankruptcy, while bankrupts B_2, B_3, and B_5 are in a financial condition typical for healthy companies. The shape of the histograms of the posterior distribution of the bankruptcy probability confirm only the conclusions relating to bankrupts B_3–B_5. The shape of the histograms in the case of companies B_1 and B_2 indicate an unclear assessment of those companies' financial condition.

Figure 5. Histograms of the posterior distribution of the probability of bankrupt companies B_1–B_5 belonging to the group of entities in a financial condition typical for companies with announced arrangement bankruptcy, and median values (marked with a black triangle) defined by means of the Bayesian logistic regression model for 2009



Source: authors' calculation using the *MCMCpack* package in R.

On the basis of the medians of the posterior distribution of the bankruptcy probability (Figure 5), only bankrupt B_1 was threatened with repeat bankruptcy. This conclusion is confirmed by the shape of this company's histogram of the posterior distribution of the bankruptcy probability. The shapes of the histograms also confirm that bankrupts B_2 and B_3 belong to the group of companies in a financial condition typical for healthy companies, while the histograms of the posterior distribution of the bankruptcy probability of companies B_4 and B_5 show an unclear assessment of their financial condition.

6. Discussion

Table 9 presents the results of the classification of the bankrupt companies under two groups: the group of entities in a financial condition typical for healthy companies ('NB' Group) and the group of entities in a financial condition typical for companies with announced arrangement bankruptcy ('B' Group), on the basis of the performed analyses and taking into account the assumptions defined in Section 3.

Table 9. Results of the classification of bankrupt companies in 'B' and 'NB' groups on the basis of the point and interval estimations of the bankruptcy probability received by applying the ML logistic regression model, and medians and the shape of the histograms of the posterior distributions of the bankruptcy probability received in the Bayesian approach

Bankrupt	2005				2006				2007				2008				2009			
	ML		*Bayes*		*ML*		*Bayes*		*ML*		*Bayes*		*ML*		*Bayes*		*ML*		*Bayes*	
	P	*CI*	*M*	*H*	*P*	*CI*	*M*	*H*	*P*	*CI*	*M*	*H*	*P*	*CI*	*M*	*H*	*P*	*CI*	*M*	*H*
B_1	B	–	B	B	B	–	B	B	B	–	B	B	B	–	B	–	B	B	B	B
B_2	B	–	B	B	B	B	B	B	B	–	B	–	B	–	NB	–	NB	NB	NB	NB
B_3	B	B	B	B	B	–	B	–	NB	NB	NB	NB	NB	NB	NB	NB	NB	NB	NB	NB
B_4	B	–	B	B	B	–	B	B	B	–	B	–	B	–	B	B	B	–	NB	–
B_5	NB	NB	NB	NB	NB	NB	NB	NB	NB	NB	NB	NB	NB	–	NB	NB	B	–	NB	–

Note. *ML* – ML logistic regression model, *Bayes* – Bayesian logistic regression model, *P* – point estimation, *CI* – 95% confidence interval, *M* – posterior median, *H* – histogram of the posterior distribution, B – bankrupt, NB – non-bankrupt.
Source: authors' calculation.

The point and interval estimations of bankruptcy probability performed on the basis of the ML logistic regression model for the year 2005 indicate a threat of repeat bankruptcy in the case of company B_3, and the similarity of the financial condition of company B_5 to a healthy company's condition (Table 9). The medians and the shape of the histograms of the posterior distributions determined on the basis of the Bayesian approach confirm the above-mentioned conclusions. Moreover, the analysis of the medians and histograms of the posterior distribution indicates that there is a threat of repeat bankruptcy for companies B_1, B_2, and B_4. These conclusions comply only with the point estimations based on the maximum likelihood method.

By means of the ML logistic regression model estimated for 2006 (Table 9), company B_2 was assigned – on the basis of the point and interval estimations – to the group of companies threatened with repeat bankruptcy, while the financial condition of company B_5 was found to be similar to that of healthy companies. The above conclusions were confirmed by the analysis of the medians and histograms of the posterior distribution defined as a result of the Bayesian approach. The use of the Bayesian approach yields a conclusion that companies B_1 and B_4 were threatened with repeat bankruptcy. This conclusion complies only with the point estimations based on the maximum likelihood method.

The results of the analysis of the point and interval estimations received on the basis of the ML logistic regression model estimated for the data for 2007 (Table 9) indicate that the financial condition of companies B_3 and B_5 is similar to that of a healthy company, which is confirmed by the analysis based on the Bayesian approach. In addition, the median and the shape of the histogram indicate a threat of repeat bankruptcy for company B_1. This suggestion is confirmed only by the point estimation based on the maximum likelihood method.

Analysing the year 2008 (Table 9), the point and interval estimations lead to a conclusion that the financial condition of company B_3 was similar to the financial condition of a healthy company. This conclusion complies with the results obtained by the Bayesian approach. The results of the Bayesian approach also suggest company B_4 is threatened with bankruptcy and the financial condition of company B_5 is similar to that of healthy companies. The point estimations of bankruptcy probability obtained by means of the ML logistic regression model leads to similar conclusions.

The results received for 2009 according to the ML logistic regression model allow a statement that company B_1 was threatened with repeat bankruptcy, while the financial condition of companies B_2 and B_3 are similar to those of healthy companies. The analysis of the medians and the shape of the histograms of the posterior distributions defined on the basis of the Bayesian approach confirmed the above findings.

The results obtained for company B_3 in the course of this study merit emphasis. In four out of five cases similar conclusions were drawn on the basis of the ML logistic regression model and the Bayesian logistic regression model. Analysing the year 2005, this company was classified in the group of entities threatened with repeat bankruptcy. In the analysis for the year 2006, the point estimation of the bankruptcy probability (ML logistic regression model) and the median of the posterior distribution (Bayesian logistic regression model) again indicated that the company was in danger of repeat bankruptcy. The interval estimation of the bankruptcy probability (ML logistic regression model) and the shape of the histogram of the posterior distribution (Bayesian logistic regression model, on the other hand, suggest an unclear assessment of the company's financial condition. From 2007 bankrupt B_3 was assigned to the group of entities with a financial condition typical for healthy companies. It should be noted that in March 2007, a court confirmed that bankrupt company B_3 fulfilled the arrangements agreed upon with its creditors in December 2005.

Company B_5 was another bankrupt for which conclusions drawn on the basis of the ML logistic regression model and the Bayesian logistic regression model were similar in the majority of cases (three out of four). In the years 2005–2007, this bankrupt was classified in the group of entities whose financial condition was typical for healthy companies. In the years 2008 and 2009, the assessment of its financial condition was unclear – the finding which was reinforced by the interval estimation obtained on the basis of the ML logistic regression model and the histogram of the posterior distribution in the Bayesian approach, in the case of 2009. A court completed the bankruptcy procedure for the bankrupt company B_5 in March 2005. With regard to that company, the court also stated that any enforcement proceedings or proceedings to secure claims conducted against the bankrupt in order to satisfy claims subject to the arrangements were discontinued, and all execution and enforcement titles became invalid.

In the case of bankrupt B_2, the conclusions made on the basis of the ML logistic regression model and the Bayesian logistic regression model were in accordance twice: in 2006, when the company was found to be threatened with repeat bankruptcy, and in 2009, when it was assigned to the group of entities in a financial condition typical for healthy companies. In May 2006, a court announced the bankruptcy procedure for this company finalized. This situation took place in April 2012 again – the court then announced bankruptcy of company B_2, including the liquidation of its assets. Taking into account the poor financial condition of bankrupt B_2 in the years 2006–2008, that is after the bankruptcy procedure was completed and its condition started improving in 2009, the reasons for this company's repeat bankruptcy in 2012 may be related to the so-called 'second wave' of the global financial crisis of 2007–2008. The impact of the macroeconomic environment on bankruptcies of companies has been discussed in literature (e.g. Platt and Platt, 2002).

In the case of bankrupts B_1 and B_4, what was mainly observed were the threats of repeat bankruptcy or an unclear financial condition. As far as company B_1 is concerned, we have not found any court records of its activities after the announcement of arrangement bankruptcy. However, in the case of bankrupt B_4, the court ruled the finalisation of the bankruptcy procedure.

7. Conclusions

In the light of the obtained results and further history of the studied bankrupt companies, the information on the financial condition of companies B_3 and B_5 prove very valuable. In the case of these bankrupts, similar results were obtained by two approaches, both based on the ML logistic regression model and the Bayesian logistic regression model. Thus, decisions made in these two companies may be a source of valuable information on how restructuring processes may improve the financial condition of companies facing insolvency problems. Nevertheless, these companies should be subject to further review, as bankruptcy may reoccur in the following years. The analysis of such cases as bankrupt B_2 provides invaluable information on recovery strategies which ended in failure and so should be avoided by enterprises undergoing bankruptcy.

In the paper, the analysis performed in the main part on the basis of the ML logistic regression model was supplemented with information yielded by the application of the Bayesian approach. This facilitated performing a broader analysis than just the assessment of the financial condition by the classification of companies into two groups, namely companies threatened with repeat bankruptcy and entities in a financial condition as that of healthy companies in a given year. In particular, the analysis of the shape of the posterior distribution of bankruptcy probability, for instance, from the perspective of probability mass distribution and the level of dispersion of distribution, in some cases enabled an observation that the financial condition of a company is not clear, despite clear assessments based on the point estimations. Such information is provided by the confidence intervals established on the basis of the ML logistic regression model, although to a smaller extent than the results received through the Bayesian approach.

In the authors' opinion, the analysis of the process of overcoming insolvency problems by companies with announced arrangement bankruptcy may provide information useful for the assessment of the success probability of different restructuring solutions, and thus valuable for future bankrupts. The logistic regression model, estimated by means of the maximum likelihood method, and the Bayesian approach prove effective in the assessment of the financial condition of companies undergoing arrangement bankruptcy in comparison to the condition of healthy companies. Models built for the years following the announcement of arrangement bankruptcy may constitute an element of the assessment system of the restructuring proposals prepared by companies under the threat of bankruptcy.

The research carried out in this paper confirms the authors' previous observations, namely that the deletion of outliers from a data set before conducting an estimation of logistic regression model parameters may improve the classification accuracy. The model estimation was based on the maximum likelihood method and the Bayesian approach. The authors' intention is to further extend the research, for example to the bootstrap method. In the case of the Bayesian approach, a scarcely informative prior distribution was adopted. In any further research, we suggest that expert knowledge should be taken into account when selecting the prior distribution.

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Determinants of spatial differentiation of labour markets in Ukraine

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Abstract. The aim of this paper is to present the differentiation of the situation in regional labour markets in Ukraine in the years 2004–2017. The analyses carried out for this purpose concern the diversity and dynamics of macroeconomic variables, such as labour productivity (measured by GDP per worker), wages, and unemployment rates.

Moreover, using panel data from the Statistical Office of Ukraine, the authors estimate the parameters of a set of equations based on increments and levels by means of the system estimator of the generalized Blundell and Bond moments method from 1998. The estimates concern the parameters of equations describing the main determinants of the increase in unemployment rates and wages for the entire Ukrainian economy – both Left- and Right-Bank Ukraine.

Keywords: unemployment rate, labour productivity, wages, Ukrainian economy, labour market
JEL: J01, J31, J64, J69, R23

1. Introduction

The aim of the analyses carried out in the paper is to assess the spatial differentiation of the situation in regional labour markets in Ukraine in the years 2004–2017. The choice of this period resulted from the availability of relevant statistical data (by region) on the website of the Ukrainian statistical office.[1]

The analyses concern the diversity and dynamics of such macroeconomic variables as labour productivity (measured by GDP per worker), wages and unemployment rates. In addition, the paper presents equation estimates of increases in unemployment rates and wages, using panel data for Ukrainian regions.

2. General characteristics of Ukrainian oblasts

According to the 1996 *Constitution of Ukraine* (Chapter IX), the country is divided into 24 oblasts (regions), the Autonomous Republic of Crimea, further referred to as 'ARK' with its capital Simferopol, and 2 special status cities: Kyiv and Sevastopol (see Map 1). Since 2014, the Autonomous Republic of Crimea and Sevastopol have been occupied by Russia, as a result of the Euromaidan of 2013/2014.

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[1] http://www.ukrstat.gov.ua/.

The Ukrainian oblasts are divided into 5 groups (for more details see e.g. Chugaievska et al., 2017, Chugaievska and Tokarski, 2018 or Tokarski et al., 2019). These groups consist of the following:

- 8 regions of Western Ukraine (Khmelnytsky, Chernivtsi, Ivano-Frankivsk, Lviv, Rivne, Ternopil, Volyn, and Zakarpattia);
- Northern Ukraine (city of Kyiv with its surroundings: Chernihiv, Kyiv region, Sumy, and Zhitomyr regions);
- 4 regions of Eastern Ukraine (Kharkiv, Donetsk, Luhansk,[2] and Zaporozhsky regions);
- South Ukraine (Autonomous Republic of Crimea, Kherson, Mikolaiv, Odessa, and Sevastopol oblasts);
- 5 oblasts of Central Ukraine (Cherkassy, Dnipropetrovsk, Kirovograd, Poltava, and Vinnytsia).

Map 1. Administrative division of Ukraine

Source: http://www.uec.com.pl/pl/ukraina/informacja_o_obwodach.

Before World War I, the Volhynia and Rivne Oblasts (situated in Volhynia) were a peripheral part of the Russian Empire, while in the interwar period this territory

[2] The Donetsk and Luhansk regions constitute an industrial area called the Donbass. The abbreviation *Донбас* (Russ. *Донбасс*) comes from the name *Донецькийвугільнийбасейн* (Russ. *Донецкийкаменноуго льныйбассейн*), or the Donetsk Coal Basin.

belonged to Poland, just like the Lviv, Ivano-Frankivsk, and Tarnopol Oblasts, which until 1918 were the north-easternmost parts of the Kingdom of Galicia and Lodomeria (part of the Austrian Empire), but in the interwar period also belonged to Poland. In the years 1849–1918, the Chernivtsi Oblast of Bukovina (the Duchy of Bukovina) belonged to the Austrian Empire, and between World War I and World War II to Romania, while the Zakarpattia Oblast was part of the Kingdom of Hungary until 1918, and in the interwar period fell under the rule of Czechoslovakia. Before World War I, the Khmelnytsky Oblast in Podolia belonged to Russia, and in the interwar period became a part of the Soviet Union. In the period between the end of World War II and Ukraine regaining independence (in 1991), all the oblasts of Western Ukraine were part of the Ukrainian Soviet Socialist Republic (part of the Soviet Union).

In short, for 200 years the regions of Western Ukraine were peripheral areas of the countries they belonged to. This situation was the reason for Western Ukraine's social and political instability, and hindered its economic development (cf. e.g. Hrycak, 2000; Serczyk, 2001; Hud, 2018).

In the 19th and 20th century, the oblasts belonging to the remaining groups of districts (in particular the districts located in Left-Bank Ukraine[3] and the coastal districts of Odessa, Mykolaiv, and Kherson) were politically and economically much more closely integrated with the Russian Empire, and later with the Soviet Union, than Western Ukraine. Therefore, their history and social, political, and economic relations were very different from that of Western Ukraine (after Hrytcak, 2000; Serczyk, 2001; Hud, 2018).

3. Differences in labour productivity, wages and unemployment rates in Ukraine

Map 2 shows the regional variation in labour productivity in Ukraine between 2004 and 2017.[4] The trajectories of this variable in groups of oblasts are illustrated in Figure 1. Map 2 and Figure 1 present the following information (see also Pustovoit, 2016; Chugaievska et al., 2017; Tokarski et al., 2019):[5]

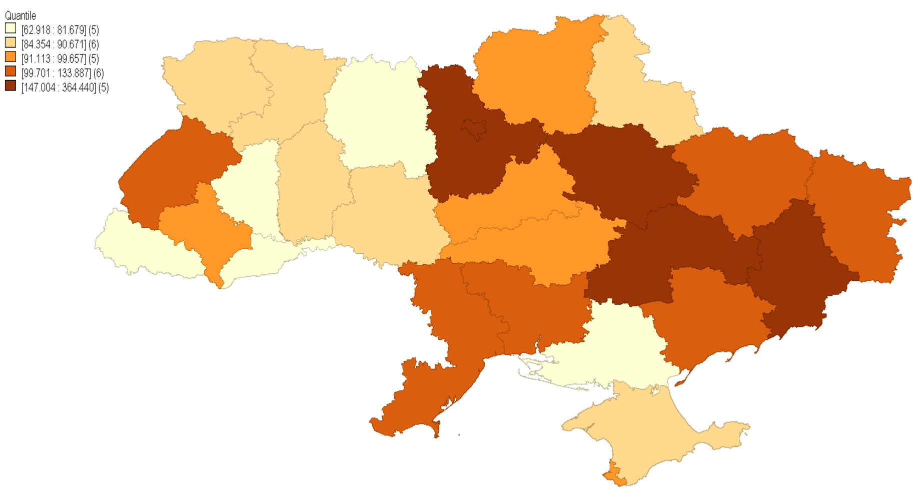
[3] Left-Bank Ukraine (Right-Bank Ukraine) is the part of Ukraine that lies to the left (right) of the largest Ukrainian river – the Dnieper.

[4] The data on the Autonomous Republic of Crimea and Sevastopol illustrated in Maps 2–4 are the average values of the analysed variables from 2004–2013 (due to the annexation of the Crimean Peninsula by Russia in 2014).

[5] All the figures analysed below, expressed in monetary units, are converted into fixed prices as of 2016. In 2016, the nominal GDP of Ukraine totaled 2,385.4 billion hryvnias, while the Polish GDP amounted to 1,861.1 billion PLN. The real GDP of Ukraine at PPP and fixed prices in 2010 was equal to 682.5 billion dollars, while the Polish GDP to 925.8 billion dollars (https://w3.unece.org/). Therefore, the dollar was (according to PPP) equal to 3.495 hryvnias or 2.010 PLN. Hence the conclusion that 100 hryvnias in 2016, including PPP, amounted to PLN 57.5.

- the highest level of labour productivity was recorded in Kyiv (364,400 hryvnias). The Dniepropetrovsk (172,100 hryvnias) and Poltava (164,900 hryvnias) Oblasts in Central Ukraine, Donetsk (157,600 hryvnias) in Eastern Ukraine and the Kiev Oblast (147,000 hryvnias) in the north of Ukraine also demonstrated a high level of labour productivity for Ukrainian conditions;
- the lowest labour productivity, i.e. below 80,000 hryvnias, was recorded in the Kherson Oblast (77,700 hryvnias) in Southern Ukraine, the Ternopil Oblast (74,700 hryvnias), the Zakarpattia Oblast (69,200 hryvnias), and in the Chernivtsi Oblast (62,900 hryvnias) in Western Ukraine;
- Left-Bank Ukraine and the Odessa and Mykolaivs'ka Oblasts are characterised by higher technical employment infrastructure and often by stronger gravitational effects than Right-Bank Ukraine (Chugaievska et al., 2017). Therefore, the level of labour productivity in these areas was generally higher than in Western Ukraine;

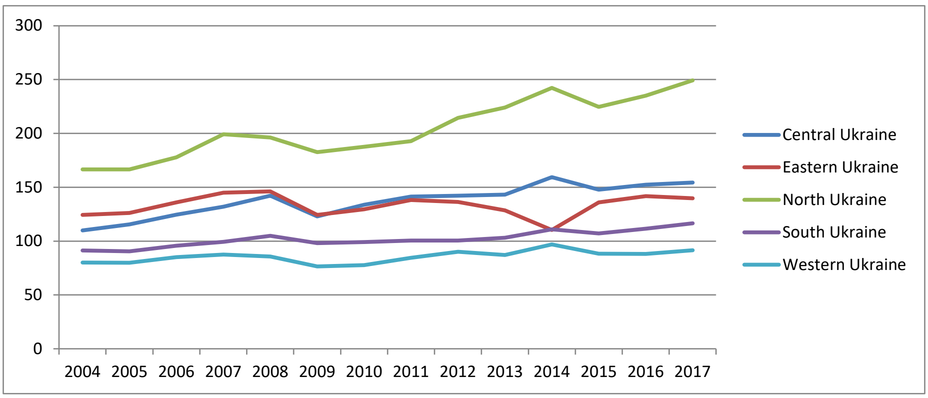
Map 2. Labour productivity in the oblasts in 2004–2017[a] (thousands of hryvnias, 2016 prices)



a For ARK and Sevastopol, 2004–2013.
Source: authors' calculations based on data from www.ukrstat.gov.ua.

- relatively large economic potential of the Ukrainian economy (measured by GDP per working person) was concentrated in Left-Bank Ukraine (Kyiv City, the Kyiv Oblast, Dnipro City, Donbas, Kharkiv, Zaporozhye), in two coastal oblasts (Odessa and Mykolaiv), and in the Lviv Oblast in Western Ukraine. Kyiv is Ukraine's centre of administration, transport, etc. Ukraine's financial services are located in the city of Dnipro, which is another centre for political, cultural, and educational life in Central Ukraine. Kharkiv, Zaporozhye and Donbass are the main regions where Ukrainian heavy industry and mining are located. The high level of economic development of the Odessa Oblast results mainly from the operations of the Odessa sea port. An important transport hub (road, railway, sea, river, and air transport) is located in Nikolayiv and the Nikolayiv Oblast in the south of Ukraine, while Lviv is by far the most economically developed city with the largest population in the western part of the country (cf. also Chugaievska and Tokarski, 2018; Chugaievska et al., 2017 or Tokarski et al., 2019);

Figure 1. Labour productivity in groups of regions (in thousands of hryvnias, prices as of 2016)



Source: authors' calculations based on data from www.ukrstat.gov.ua.

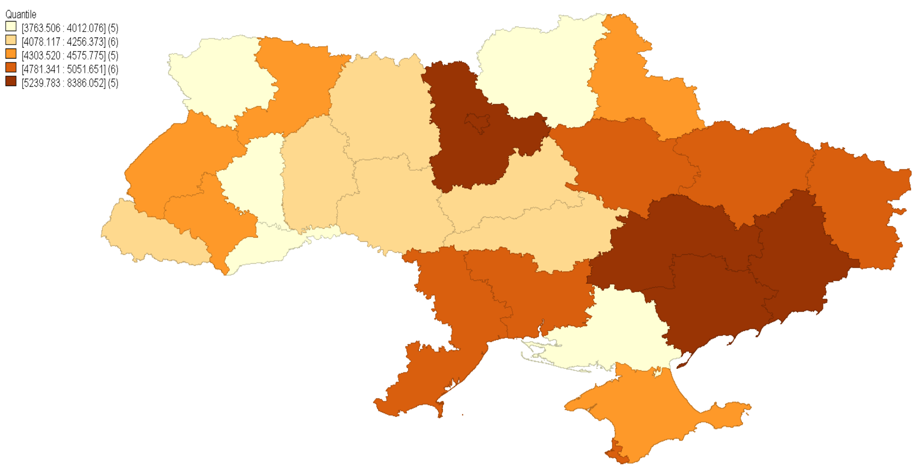
- all groups of oblasts (except Eastern Ukraine) show similar trends in labour productivity. Between 2004 and 2008, the value of this variable increased, but then decreased significantly in 2009. This decline was the result of both the global financial crisis and the gas conflict with Russia. Between 2010 and 2014 (i.e. until the Euromaidan), GDP per worker started to grow again. In 2015, labour productivity decreased, while an increase occurred in the years 2016–2017, as Ukraine recorded its first symptoms of economic recovery after the crisis caused by political perturbations following the Euromaidan (in 2016 Ukrainian GDP grew by 2.4% and in 2017 by 2.5%);
- the regions of Northern Ukraine, where the value of labour productivity increased from 166.6 thousand hryvnias in 2004 to 249.2 thousand hryvnias in 2017, noted the highest levels of GDP per employee. In Eastern Ukraine, labour productivity increased from 124.4 thousand hryvnias to 139.7 thousand hryvnias, in Central Ukraine from 110.0 thousand hryvnias to 154.4 thousand hryvnias, in the south

from 91.3 thousand hryvnias to 116.6 thousand hryvnias, and in the west from 80.0 thousand hryvnias to 91.5 thousand hryvnias. This variable grew fastest in the most developed region – Northern Ukraine (3.1% of the annual average), and the slowest in the industrial and mining regions of Eastern Ukraine (0.9%).

Map 3 shows the geographical variance of wage distribution in Ukraine, while Figure 2 shows the trajectories of this variable in groups of oblasts in the years 2004–2015. The following conclusions can be drawn from the map and figure (cf. also e.g. Bolińska and Gomółka, 2018):

- as in the case of labour productivity, the highest wages were recorded in the capital city of Kyiv (on average 8,386.05 hryvnias in the years 2004–2017). The Donetsk (5,990.33 hryvnias), Dnipropetrovsk (5,481.32 hryvnias), Kiev (5,330.82 hryvnias), and Zaporozhsky (5,239.78 hryvnias) oblasts also noted high values of this variable;

Map 3. Wages in oblasts 2004–2017[a] (in hryvnias, prices as of 2016)

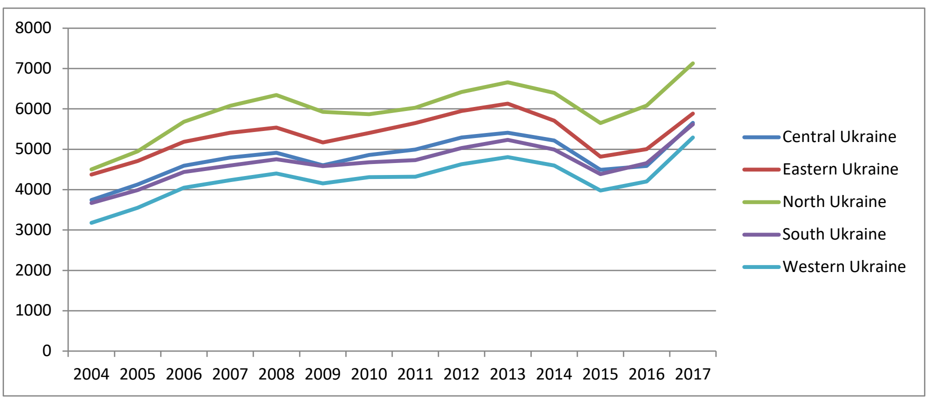


a For the Autonomous Republic of Crimea and Sevastopol, 2004–2013.
Source: authors' calculations based on data from www.ukrstat.gov.ua.

- the lowest wages were observed in the Chernivtsi (4,012.08 hryvnias), Volyn (4,005.88 hryvnias), and Ternopil (3,763.51 hryvnias) oblasts in Western Ukraine, in Kherson (4,009.69 hryvnias) in Southern Ukraine, and in Chernihiv (4,000.78 hryvnias) in the north of Ukraine;

- spatial wage differentials in Ukraine overlapped with productivity differences to a large extent. The correlation coefficient between these variables amounted to 0.946, while the correlation coefficient between wages and unemployment rates discussed below, amounted to 0.586;
- wage trajectories in groups of oblasts were similar in shape to both the GDP and labour productivity trajectories, as labour productivity had a significant impact on wage levels in Ukraine;
- in Northern Ukraine, wages increased from 4,499.27 hryvnias in 2004 to 7,130.00 hryvnias in 2017, in Eastern Ukraine from 4,373.79 hryvnias to 5,881.58 hryvnias, in Central Ukraine from 3,741.65 hryvnias to 5,654.84 hryvnias, in Southern Ukraine from 3,669.01 hryvnias to 5,614.69 hryvnias, and in the poorest western regions of Ukraine from 3,176.35 hryvnias to 5,291.34 hryvnias. Therefore, the highest average annual wage dynamics were observed in Western Ukraine (4.0%) and the lowest in Eastern Ukraine (2.3%).

Figure 2. Wages and salaries in groups of regions (in hryvnias, prices as of 2016)



Source: authors' calculations based on data from www.ukrstat.gov.ua.

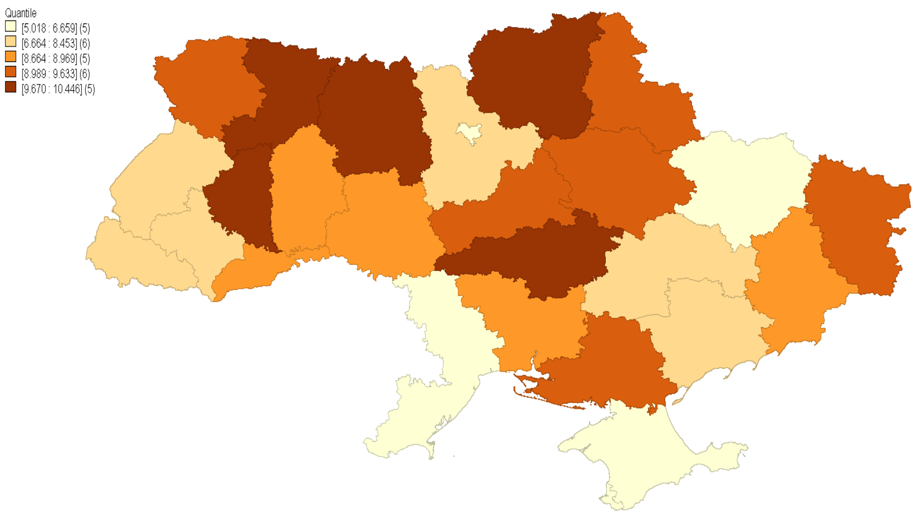
Map 4 illustrates the regional differentiation of unemployment rates in Ukrainian oblasts (2004–2017), while Figure 3 illustrates the trajectories of this variable in groups of oblasts. The following conclusions can be drawn from the map and the chart presented below (see also e.g. Lysiuk and Kaflevska, 2012; Paniuk, 2013; Homiak, 2015; Jarova, 2015; Tokarski et al., 2019):

- the lowest average unemployment rates in the studied period were observed in two cities with special status (Sevastopol 5.0% and Kyiv 5.3%), in the Crimean Autonomous Republic (5.7%), and the periphery of Odessa (5.9%) in the south of Ukraine. The highest unemployment was recorded in the Zhitomyr region

(10.1%) and in the Ternopil region (10.3%) in the north of Ukraine, and the Rivne region (10.4%) in Western Ukraine. High unemployment rates were also observed in the Chernihiv Oblast (9.8%) in the north and in the Kirovograd Oblast (9.7%) in Central Ukraine;

- the geographical variation in unemployment rates in Ukraine partly coincided with the geographical variation in labour productivity and wages, in the sense that, usually, the higher the productivity or wages, the lower the unemployment rates. The correlation coefficient between unemployment and labour productivity was −0.499, and between unemployment rates and wages −0.586;

Map 4. Unemployment rates in groups of oblasts 2004–2017[a] (%)

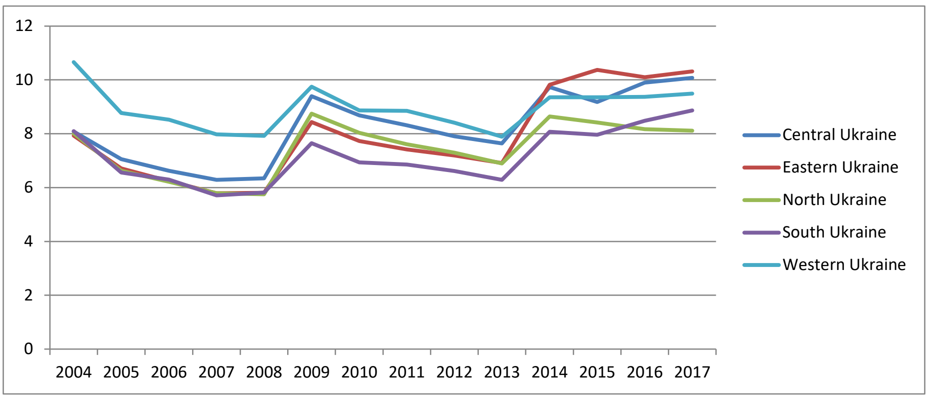


a For the Autonomous Republic of Crimea and Sevastopol, 2004–2013.
Source: authors' calculations based on data from www.ukrstat.gov.ua.

- the years 2004–2008 marked a good period for the Ukrainian economy as GDP grew rapidly, which translated into an increase in employment and a decrease in unemployment in all groups of oblasts. At that time, unemployment fell fastest in Western Ukraine (down by 2.7 percentage points) and slowest in Central Ukraine (1.7 percentage point). As a result, the difference between the group of oblasts with the highest unemployment (Western Ukraine) and the group with the lowest (Eastern Ukraine) decreased from 2.7 percentage points in 2004 to 2.1 percentage points in 2008;

- the global financial crisis, combined with the Russian-Ukrainian gas conflict, brought about a one-year recession, which also resulted in a significant increase in unemployment in all groups of districts. At that time, it grew fastest in Central Ukraine (by 3.1 percentage points) and slowest in the south and west of Ukraine (by 1.8 percentage points);

Figure 3. Unemployment rates in groups of oblasts (%)

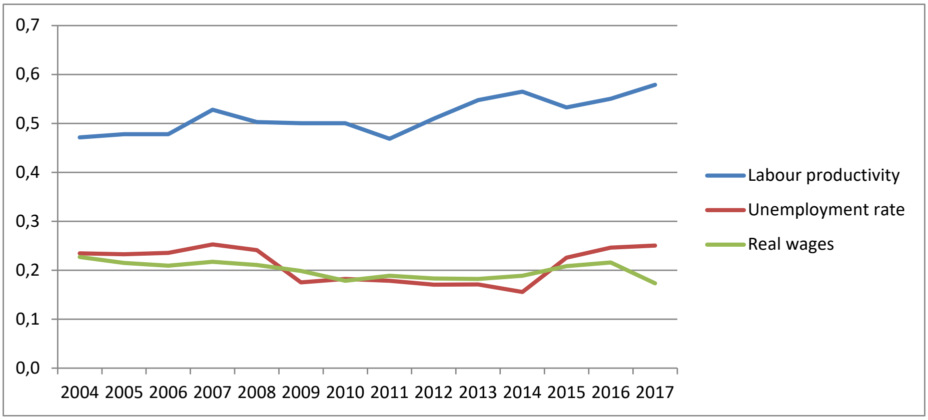


Source: authors' calculations based on data from www.ukrstat.gov.ua.

- the economic growth in Ukraine between 2010 and 2013 caused the unemployment rate to fall in all groups of oblasts. The largest falls in unemployment were recorded in Western Ukraine (1.9 percentage point) and the smallest in Southern Ukraine (1.4 percentage point);
- the economic and political-military crisis following the Euromaidan caused a surge in unemployment in all groups of Ukrainian oblasts. It increased from 7.9% in 2013 to 9.4% in 2014 in Western Ukraine, from 7.6% to 9.7% in Central Ukraine, from 6.9% to 9.8% in Eastern Ukraine, from 6.9% to 8.6% in Northern Ukraine and from 6.3% to 8.6% in Southern Ukraine;
- in the years 2015–2017, unemployment in all groups of Ukrainian oblasts stabilised, except for the regions of Northern and Southern Ukraine. In the north it started to grow slightly, and in the south, to fall slightly.

Figure 4 presents the coefficients of variation (quotient of standard deviation and unweighted arithmetic mean) of the macroeconomic variables analysed in Ukrainian oblasts in the years 2004–2017. The conclusion is that labour productivity was much more regionally varied than wages and unemployment rates. Moreover, the regional differentiation in labour productivity in Ukraine was on an upward trend, while the wage and unemployment rate differentials were similar and fairly stable over time.

Figure 4. Labour productivity, wage, and unemployment rate variability rates in Ukraine



Source: authors' calculations based on data from www.ukrstat.gov.ua.

4. Estimated parameters of the equations for changes in unemployment rates and wages

In order to calculate the main determinants of the increase in unemployment rates, the definition of the unemployment rate provided by the following formula was applied (see also Tokarski, 2005):

$$u_{it} = \frac{U_{it}}{U_{it} + L_{it}} = 1\frac{L_{it}}{N_{it}} \tag{1}$$

where:
U_{it} – the number of unemployed,
L_{it} – the number of employed,
N_{it} – the supply of labour in the *i*-th province of the year *t*.

Equation (1) allows the increase in unemployment rates to depend on the level of the unemployment rate from the previous period and the rate of product growth. For this purpose, one can differentiate this equation to obtain the following relationship:

$$u'_{it} = \frac{L'_{it} \cdot N_{it} - L_{it} \cdot N'_{it}}{N_{it}^2} = \frac{L_{it}}{N_{it}} \cdot \left[\frac{N'_{it}}{N_{it}} - \frac{L'_{it}}{N_{it}}\right]$$

Hence, and by the definition of the unemployment rate (1), the increase in the unemployment rate can be written as follows:

$$u'_{it} = (1 - u_{it}) \cdot \left[\frac{N'_{it}}{N_{it}} - \frac{L'_{it}}{N_{it}}\right] \tag{2}$$

Assuming, additionally, that the growth rate of the number of employed $\frac{L'_{it}}{N_{it}}$ is an increasing function of the product growth rate *g*, and by using the relationship (2), we arrive at the following equation for the increase in the unemployment rate (see also Dykas et al., 2013):

$$u'_{it} = (1 - u_{it}) \cdot \left[\frac{N'_{it}}{N_{it}} - f(g)\right] \tag{3}$$

where:

$$\frac{L'}{L} = f(g), f'(g) > 0$$

Equation (3) demonstrates that, firstly, the increase in the unemployment rate is a decreasing function of the product growth rate g, and secondly, that if the labour supply growth rate is greater (smaller) than the growth rate of the number of employed, then the increase in the unemployment rate is a decreasing (increasing) function of the unemployment rate.

While analysing factors which determine wages, the following reasoning can be applied, which is a combination of the Solow 1979 efficiency wage model and the neoclassical economic growth model of Solow 1956 and its generalizations (Tokarski, 2005 or Dykas and Misiak, 2014). In the classic Solow 1979 efficiency wage model, an enterprise operating on the market aims to maximize the profit function described by the following formula:

$$\pi(w_{it}, L_{it}) = F(\varepsilon(w_{it}) \cdot L_{it}) - w_{it} \tag{4}$$

where:

w_{it}, L_{it} – wages and number of employees in the *i*-th province in period *t*,

$\varepsilon(w_{it})$ – the efficiency of a typical employee, which is assumed to be an increasing wages function ($\varepsilon'(w_{it}) > 0$),

$F(\varepsilon(w_{it}) \cdot L_{it})$ – a neoclassical production function that describes the relationship between the so-called units of effective work (the product of the effectiveness of a typical employee $\varepsilon(w_{it})$ and the number of employees L_{it}), and income.

The conditions necessary to maximise the profit function $\pi(w_{it}, L_{it})$ are equal to

$$\frac{d\varepsilon(w_{it})}{dw_{it}} \cdot \frac{w_{it}}{\varepsilon(w_{it})} = 1 \tag{5}$$

In the next stage one can use the efficiency function of a typical employee with the given formula:

$$\varepsilon(w_{it}) = \left(\frac{w_{it} - x_{it}}{x_{it}}\right)^{\alpha} \tag{6}$$

where $\alpha \in (0, 1)$, while x_{it} is the minimum wage that a typical employee is able to accept (the wage x_{it} is sometimes called the threshold wage).

In addition, it is assumed that the threshold wage is described by the following equation:

$$x_{it} = (1 - a \cdot u_{it}) \cdot w_t \tag{7}$$

where:
u_{it} – the unemployment rate in the i-th province in period t,
w_{it} – the average wage in period t.

Equations (6) and (7) show that the effectiveness of a typical employee is an increasing function of a relative wage gap in the i-th labour market from the threshold wage operating in this market and the threshold wage is an increasing function of the average wage and a decreasing function of the unemployment rate. The following relationship between relative wages and the unemployment rate derives from Equations (5–9):

$$\widetilde{w}_{it} = \frac{1}{1-\alpha} - \frac{a}{1-\alpha} u_{it} \tag{8}$$

where:

$$\widetilde{w}_{it} = \frac{w_{it}}{\overline{w}_{it}} u_{it}$$

and $\overline{w}_{it}$ is the average level of wages in the i-th market in year t.

Moreover, it can be assumed that wages are ultimately in line with the marginal product of labour (as is the case with the Solow 1956 economic growth models and its generalizations). Therefore, with the power production function of the Cobb-Douglas type, wages are proportional to work efficiency. Taking into account the above considerations, Equation (8) can be extended to the following equation:

$$\widetilde{w}_{it} = \alpha_0 - \alpha_1 \cdot u_{it} + \alpha_2 \cdot y_{it} \tag{9}$$

In view of this, the basic determinants of relative wages are labour productivity and unemployment rate.

Hence, the parameters of the following equations were estimated:[6]

$$\Delta u_{it} = \beta_0 - \beta_1 u_{it-1} + \beta_2 d_{\Delta u} u_{it-1} + \beta_3 \Delta \ln Y_{it} \tag{10}$$

and

$$\widetilde{w}_{it} = \alpha_0 - \alpha_1 u_{it} + \alpha_2 \ln y_{it} \tag{11}$$

where u_{it} stands for the unemployment rate in oblast *i* in year *t*, $d\Delta u$ is a zero-one variable taking the value of 1 when the unemployment rate in circumference in year *t* increased, and taking the value of zero in other cases (this variable makes it possible to distinguish trends in changes of the unemployment rate due to the direction of changes), $\Delta \ln Y_{it}$ is the real GDP growth rate in oblast *i* in year *t*, w_{it} indicates real average gross wages in oblast *i* in year *t*, and y_{it} signifies the level of labour productivity in oblast *i* in year *t*.

Equations (10–11) were estimated using the system estimator of the generalized moments method (SGMM). The idea of SGMM is to estimate the system of equations on both increments and levels. The instruments for explanatory variables in equations at levels are the delayed first increments of these variables.

The results of the estimation of parameters of Equations (10–11) for Ukraine, Left-Bank Ukraine, and Right-Bank Ukraine are presented in Tables 1–2.

The following conclusions can be drawn from the estimations presented in Table 1:

- the estimated signs of the parameters of the equation for the increase in unemployment rates are consistent with the theory of macroeconomics (with the exception of the parameter determining the impact of real GDP growth rate, which was statistically insignificant for Right-Bank Ukraine);
- analysing the estimates summarised in Table 1, a large asymmetry of the impact of unemployment rates from the previous period on the increase in unemployment rates could be observed, depending on whether previous unemployment rates increased or decreased in each of the discussed regions of Ukraine. The estimated parameter was higher (in terms of module) when unemployment rates in the previous period indicated a downward trend;

[6] For all parameters in Equations (10–11), except constants, which are implicitly assumed to be positive values.

Table 1. Estimation of the parameters of the unemployment rate growth equation for Ukraine – Left-Bank Ukraine and Right-Bank Ukraine

Explanatory variable	Ukraine		Left-Bank Ukraine		Right-Bank Ukraine	
	Coeff.	*t*-Stat	Coeff.	*t*-Stat	Coeff.	*t*-Stat
Δu_{it-1}	0.0671	1.75 (0.092)	0.0594	1.25 (0.232)	0.0108	0.23 (0.821)
constant	0.0296	6.12 (0.000)	0.0274	4.28 (0.001)	0.0301	4.20 (0.001)
u_{it-1}	−0.4366	6.12 (0.000)	−0.4319	4.42 (0.001)	−0.4176	−5.55 (0.000)
$d_{\Delta u} u_{it-1}$	0.2754	21.02 (0.000)	0.2877	8.66 (0.000)	0.2529	17.11 (0.000)
$\Delta \ln Y_{it}$	−0.0337	−5.03 (0.000)	−0.0427	−4.07 (0.001)	−0.0161	−1.61 (0.134)
Test AR(1)	−3.90 (0.000)		−2.67 (0.008)		−2.99 (0.003)	
Test AR(2)	0.18 (0.854)		−0.04 (0.966)		0.35 (0.729)	
Hansen test	0.66 (0.882)		2.39 (0.495)		0.48 (0.923)	
F test	146.04 (0.000)		145.12 (0.000)		104.56 (0.000)	
Number of observations	316		160		156	
Number of instruments	8		8		8	

Note. The levels of statistical significance are given in brackets.
Source: authors' calculations based on data from www.ukrstat.gov.ua.

- the unemployment rate from the previous period (with the downward trend) had the strongest impact on the increase in the current unemployment rate for the entire Ukrainian economy, where each subsequent percentage point of the decrease in unemployment translated into a fall in current unemployment of about 0.44 percent. On the other hand, in the situation of rising unemployment rates in earlier periods, the strongest impact on the increase in current unemployment was observed in Left-Bank Ukraine, where each subsequent percentage point of increase in the unemployment rate caused an increase in current unemployment by about 0.29 percent;
- an increase in real GDP growth rate of 1 percentage point caused a decrease in unemployment rates in both the entire Ukrainian economy and in Left-Bank Ukraine of 0.034 and 0.043 percentage points, respectively. The estimated parameter determining the impact of the real GDP growth rate on the increase in current unemployment for Right-Bank Ukraine proved statistically insignificant;
- the Hansen test statistics and significance levels obtained for the studied regions in Ukraine do not give grounds to reject the zero hypothesis that all instruments in the model are not correlated with the random component. The Arellano-Bond test values for AR(1) and AR(2) (see also Arellano and Bond 1991) were also satis-

factory, which indicates that in all estimation variants negative, statistically significant first order autocorrelation and statistically insignificant second order autocorrelation were obtained. This demonstrates the compatibility and effectiveness of the applied estimators.

The analysis of the results of the estimation of the relative wage equations (presented in Table 2) enables the following conclusions to be drawn:

- the estimated parameters of the wage model for the Ukrainian economy are statistically significant in the case of Left-Bank Ukraine. The parameter determining the impact of labour productivity on wages was statistically insignificant for Right-Bank Ukraine. The parameter describing the impact of the unemployment rate on wages also turned out to be statistically insignificant. Moreover, the scope of influence of independent variables on the level of wages proved consistent with the theory;
- the estimated parameters determining the level of impact of unemployment rates on relative wages for the entire Ukrainian economy were (in terms of module) higher than the estimates for Left-Bank Ukraine. In addition, the estimated parameter demonstrating the impact of labour productivity on relative wages for Right-Bank Ukraine was over 40% higher than for the entire Ukrainian economy.

Table 2. Estimates of parameters of the relative wage equation for Ukraine, Left-Bank Ukraine and Right-Bank Ukraine

Explanatory variable	Ukraine		Left-Bank Ukraine		Right-Bank Ukraine	
	Coeff.	*t*-Stat	Coeff.	*t*-Stat	Coeff.	*t*-Stat
$\widetilde{w}_{it-1}$	0.9222	12.45 (0.000)	0.9590	10.83 (0.000)	0.8547	5.75 (0.000)
constant	−0.9159	−1.31 (0.191)	−0.1317	−2.10 (0.036)	−0.1305	−1.08 (0.281)
$\ln(y_{it})$	0.0421	2.92 (0.004)	0.0414	1.90 (0.048)	0.0603	2.00 (0.046)
u_{it}	−0.322	−3.63 (0.000)	−0.2976	−2.42 (0.015)	−0.0604	−0.29 (0.769)
Test AR (1)	−1.74 (0.082)		−1.51 (0.130)		−1.96 (0.050)	
Test AR (2)	−1.46 (0.144)		−1.13 (0.258)		−1.35 (0.176)	
Hansen test	3.11 (0.375)		2.86 (0.413)		4.51 (0.211)	
F test	104.82 (0.000)		156.36 (0.000)		146.92 (0.000)	
Number of observations	343		174		169	
Number of instruments	7		7		7	

Note. The levels of statistical significance are in brackets.

Source: authors' calculations based on data from www.ukrstat.gov.ua.

5. Conclusions

Geographical diversity of economic development and, consequently, regional labour markets in Ukraine are largely determined by historical factors. The most developed parts of the country are the largest cities (Kyiv, Dnipro, Kharkiv, Donetsk, Lugansk, Zaporozhye, Odessa, Mikolaiv, and Lviv). They have developed either their service sectors (Kyiv, Dnipro, Odessa, Mikolaiv, Lviv) or industrial sectors (Kharkiv, Zaporozhye, Donbas). All these cities (with the exception of Lviv) are located on the left bank of Ukraine or on the Black Sea. In other words, they are located in areas which, for over 200 years, were much more closely integrated with the Russian Empire, and later with the Soviet Union, than with the rest of Ukraine.

The trajectories of basic macroeconomic variables influencing the situation in regional labour markets depended both on the business cycle and the political cycle. Until 2008, the Ukrainian economy was developing rapidly as a result of market-oriented reforms undertaken at the beginning of the 21st century. Then came recession, caused by the global financial crisis and the gas conflict with Russia, which led to a fall in GDP, labour productivity, and wages, and a rise in unemployment. After that, the Ukrainian economy returned onto a path of economic growth, which lasted until the outbreak of Euromaidan. The Russian annexation of the Crimean Peninsula and the military conflict with pro-Russian separatists in Donbass destabilised Ukraine, which caused the country to plunge into deep recession, from which the Ukrainian economy began to emerge only in 2016.

The regional variation in labour productivity in Ukraine was greater in the studied period than the wage and unemployment differences. The reason behind this situation is threefold. Firstly, the differentiation in labour productivity resulted mainly from differences in employment infrastructure and the extent of gravity effects (which is largely dependent on the effect of the centuries-old urban network, cf. Chugaievska et al., 2017). Secondly, the public sector of the Ukrainian economy was characterised by hidden unemployment, which added to the smaller geographical variation in unemployment rates. Unemployment was much more evenly distributed geographically than employment infrastructure or labour productivity. Thirdly, the wage gap in Ukraine was much smaller than the productivity gap, because, as in the case of Poland (cf. e.g. Trojak and Tokarski, 2013), wages in the public sector were quite uniform throughout the country.

The estimates of the parameters of the equation for the increase in unemployment rates show that these increases were most strongly influenced by the values of unemployment rates from the previous period when these rates were on an upward trend for Left-Bank Ukraine. In the situation where the unemployment rates were on a downward trend, then unemployment rates from the previous period had the strongest impact on the current increase in unemployment for the entire Ukrainian economy. The GDP growth rate had a stronger impact on the increase in unemployment in Left-Bank Ukraine, while the parameter describing the impact of the GDP growth rate on the increase in the unemployment rate for Right-Bank Ukraine proved statistically insignificant. In the case of relative wage equations, the impact of labour productivity on relative wages in Left-Bank Ukraine and the entire economy was symmetrical and about 20% lower than in Right-Bank Ukraine. Moreover, the unemployment rate had a stronger impact on relative wages in the entire economy than on Left-Bank Ukraine, while the parameter determining the impact of the unemployment rate on the relative wages of Right-Bank Ukraine proved statistically insignificant. To sum up, the weaker impact of GDP growth rates on increases in unemployment rates and labour productivity on wages can be explained by the existence of hidden unemployment in the public sector of the economy and low regional diversity of wages in this sector (see also Chugaievska and Tokarski, 2018).

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A proposal for perception measurement on a linguistic scale coded with unconventional fuzzy numbers

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Abstract. The aim of this paper is to formulate a new proposal for perception measurement on a linguistic scale coded with fuzzy numbers. Additionally, an attempt is made to show the assessment process of the adequacy of a linguistic scale. The basis for the proposal is the discussion of issues related to the ambiguity of the results of measurements made by means of a subjective type of measurement scales. The proposed assessment technique is relevant when the results of the measurement based on a linguistic scale are coded with numerical equivalents in the form of e.g. unconventional fuzzy numbers.

The issue the subjective perception of the products' quality illustrates the objectivity level of measurement results. Subjective perception is measured with a specially designed IT tool allowing the respondent to determine all the characteristics of the resulting fuzzy numbers. The scale adequacy assessment tool is based on the Item Response Theory, and particulary so on the model devised by Georg Rasch.

The measurement of socio-economic phenomena, including material and subjective well-being of households, the quality of households' durable goods, and the assessment of the quality of goods available on the market requires special tools. It seems that one of the most useful and powerful tools for the measurement of socio-economic phenomena is a linguistic scale. The problematic issue in the analysis presented in the paper is coding verbal terms with their numerical equivalents.

Keywords: measurement, measurement scale, measurement scale adequacy, Item Response Theory, the Rasch model

JEL: C12, C52, C81, C82, C83

1. The introduction and motivation

The problem examined in the paper is the measurement of households' subjective perception of wellbeing. It is challenging to measure concepts such as wellbeing and its perception directly with a numerical scale, as both these notions are of qualitative nature (Tov and Diener, 2009). The existing measurement tools, such as ones based on the Likert scale, the 'divide 100 points' scale, semantic scales or benefit structure analysis do not address the heterogeneity of perception and subjectivity precisely enough. In other words, the existing measurement tools fail to recognise the variety and heterogeneity of respondents' statements (Walesiak and Gatnar, 2009). The research presented in this paper is aimed at designing new, improved measurement techniques appropriate for this type of socio-economic phenomena. The problem of household wellbeing belongs to a wider class of socioeconomic problems, where a subjective perception is a decisive factor for the measurement

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results and, as such, requires a special means of measurement (Michalos, 2014; Fattore et al., 2011). One of the most important areas of subjective perception measurement is the quality and dignity of life.[1]

The importance of the results of the measurement of subjective consumer preferences for business could be described by e.g. marketing managers or decision-makers, who have benefitted from the valuable information yielded by this kind of measurement. The issue of subjectivity of consumers' perception is equally important with regard to new, innovative durable goods. Dynamic technological changes nowadays make it impossible for consumers to form informed opinions about such products based on objective technical and technological facts, so instead, they form their judgements on the basis of subjective impressions.

The definition of household wellbeing distinguishes among several approaches (ONS, 2019a, 2019b; Saisana, 2014).[2] The phenomenon of the households' subjective perception of wellbeing requires extensive discussion, which was initiated in the work by Diener et al. (2018). At this point it should be clarified, though, that this paper will not deal with the theoretical concept of the subjective perception of wellbeing, but will focus on the early stages of introducing improved measurement techniques of the subjective perception of qualitative phenomena, as seen from the methodological point of view.

Since it is difficult to quantify perceptions on a metric scale, the researcher may request the respondents to use verbal, linguistic phrases to describe their perception of the object of interest. Linguistic variables seem to be one of the most promising techniques for measuring socioeconomic phenomena. A linguistic variable is one whose values are presented in the form of verbal categories, to which, in turn, numerical codes are assigned. In the measurement practice, linguistic terms are used to measure the status of the selected socio-economic phenomena, which include the subjective perception of welfare, the subjective assessment of the quality of a household's durable goods, etc. Wherever subjectivity is involved, linguistic variables are convenient and intuitive means of assessing perceptions or preferences, since their values are defined as verbal categories. Therefore, linguistic variables make it possible to quantify the criteria for phenomena which by nature are categorical and potentially perceived differently by every respondent (*FisPro…*, 2018, p. 45–48). The usability of linguistic variables in the studies of socio-economic phenomena is moreover indicated in literature, for example in the work by Schnorr-Bäcker (2018).

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[1] OECD Better Life Index, OECD, Paris, www.oecdbetterlifeindex.org (accessed 20.11.2019).

[2] Quality of Life Research, selected issues of an Official Journal of the International Society of Quality of Life Research (*International Journal of Quality of Life Aspects of Treatment, Care and Rehabilitation*).

As was mentioned before, it is difficult to interpret results obtained by means of a subjective type of measurement scales unequivocally, and this is how the method of quantifying linguistic expressions became the subject of this paper. It is important to remember here that unless fuzzy numbers are applied to encoding verbal statements, it is difficult to formulate clear recommendations on how to determine the domain of fuzzy numbers for a verbal variable. The author's experience relating to the measurement of socio-economic phenomena (including the material wealth of households determined by their possession of durable goods) leads to the conclusion that coding linguistic variables, i.e. verbal statements, into numerical equivalents is most effective when performed with the use of the so-called unconventional fuzzy numbers, i.e. numbers which have an uneven length, are unbalanced and are of an overlapping shape (Arguelles Mendez, 2016; Roubens and Vincke, 1988).

The application of linguistic variables is beneficial for the respondent, but the researchers are left with a difficult task of coding verbal statements into numerical equivalents adequately and choosing the analytical techniques and the base for inference. Using fuzzy numbers for this purpose is one of the possible approaches (Zalnezhad and Sarhan, 2014; Kacprzyk and Roubens, 1988; Zadeh, 1975). The aim of this paper is to estimate the degree of adequacy and precision of linguistic variables used as measurement tools for subjective assessment. In this context, the latent trait[3] models seem to be promising instruments of the assessment of scale adequacy. They are designed to measure the underlying ability or trait, which the test result indicates, rather than measuring the performance *per se*. Another argument in favour of latent test models is that the structure of the test is sample-free. The results are independent of the measurement scheme which generated them. The method of latent trait models was developed in the 1950s, but due to the lack of specialised computer software, it could not be used in empirical research and thus for a long time it remained a theoretical concept with no practical applicability.

The Item Response Theory Models seem to be an adequate tool for the assessment of the relevance and precision of measurement scales based on linguistic variables. Measuring and assessing the scale adequacy helps to improve the coding quality in the process of replacing verbal statements with responses in the form of unconventional fuzzy numbers. The Item Response Theory[4] (IRT) is a theoretical system of models, including probabilistic ones, applicable to the analyses and evaluation of measurement scales. The technique is associated with the name of one of the authors, Georg Rasch. The scaling used in the IRT models assumes that anyone who can re-

[3] Trait is a distinguishing feature of a person's character. Often, in literature on the subject, a trait is called respondent characteristic or respondent ability.

[4] The adopted convention is that names which can be used in an abbreviated form are written with capital initials, e.g. Item Response Theory could also be referred to as IRT.

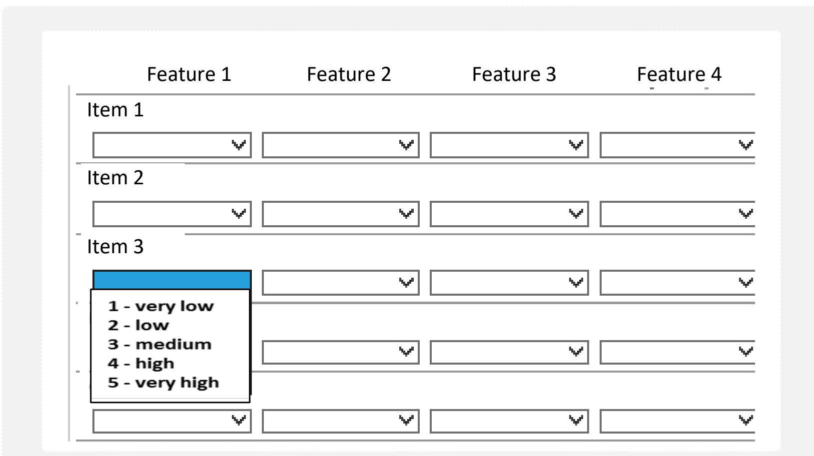
spond to statements of high difficulty will be able to respond to statements of low difficulty, too. The Item Response Theory models for the assessment of test items and questions belong to a group of models gradually developing in social research and applicable to the following areas: psychology, education (for example in the PISA study), medicine and marketing. The family of IRT models is rooted in theories devised by L. Guttman and R. Mokken, who introduced non-parametric probabilisation of the Guttman scalogram (Guttman, 1944; Guttman et al., 1950; Hofmann, 1979; Abdi, 2010; Mokken, 1971; Sijtsma and Ark, 2017; Ark, 2012; Wind, 2017; Watson et al., 2018).

In order to measure the subjective perception of socioeconomic phenomena effectively, the best solution seems to be, as mentioned before, the application of a linguistic scale with verbal categories used for determining the assessment results of a group of respondents. It has also been noted that numerous characteristics of socio-economic phenomena are inherently qualitative, therefore conventional, quanti-tative measurement tools fail to fully overcome problems which often occur in the process of measuring the perceptions or attitudes of respondents. The aforementioned problems lie in the fact that the researcher attempts to quantify characteristics which are either immeasurable (on metric scales) or hidden. For this reason, an alternative approach needs to be applied, involving the use of verbal, linguistic phrases to describe such characteristics as attitudes towards or perception of the phenomenon of interest. Verbal, linguistic phrases that attempt to capture and explain the differences in the individual respondent's assessments of those phenomena constitute the measuring technique recommended in socio-economic analyses (European Commission, 2017; Zamri and Abdullah, 2014).

2. Conceptual framework. Measurement method. Linguistic form of characteristics' level determination

Although the advantages of using linguistic variables, which are intuitive and convenient for respondents to express their judgements opinions or preferences, have already been discussed, they cannot be overestimated. A linguistic variable is a form of a characteristic whose values are determined using a verbal category. The linguistic form of a characteristic is referred to as a linguistic feature. It may be used by the respondent for the description of the subjectively perceived level of the measured characteristic of a socio-economic phenomenon. Linguistic features have values defined as verbal categories. But, as was mentioned before, using linguistic features poses serious problems for the researcher, who has to face the challenging task of adequately encoding verbal statements into numerical values. Along with the advantages of linguistic variables comes a basic conceptual problem – phenomena are inherently descriptive and as such, having the potential of being understood differently by different respondents. One of the possible solutions to this problem is to employ a linguistic description, which might further be translated, i.e. coded, into some form of numerical values. The algorithm of the procedure involves the use of linguistic variables to determine a respondent's assessment of a given phenomenon by indicating one of the verbal levels of the linguistic variable. Subsequently, linguistic variable levels are assigned to their numerical equivalents, i.e. coded usually into some forms of fuzzy numbers. Both steps are performed by the respondent. In the first step, respondents select verbal categories.

Figure 1. Respondents' choice of verbal categories included in the assessed items



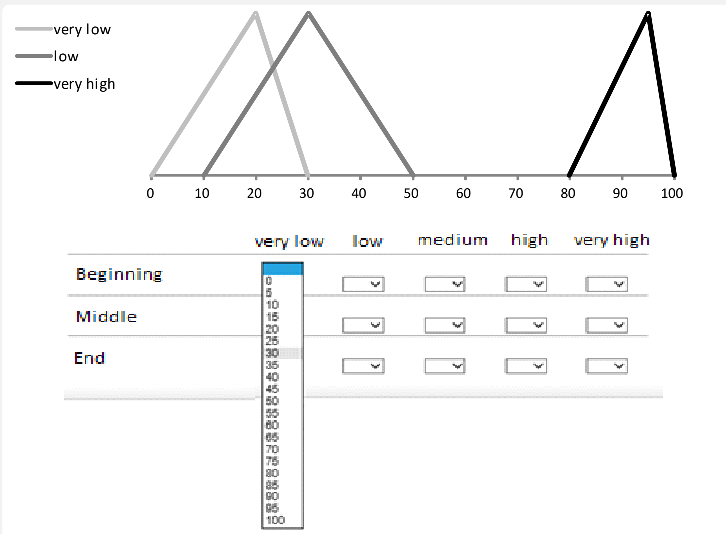
Source: author's work.

In the second step, respondents perform the coding of verbal categories with numerical values. Figure 1 presents an example of a computer screen view for respondents with instructions on how to define the lower, medium, and upper value. The respondent may see the immediate graphical illustration of coding verbal categories in the form of triangles. The outcome of step 1 and 2 of the measurement procedure applied to all respondents is obtaining the measurement results in the form of fuzzy numbers. Such numbers take an unconventional form. Figure 2 illustrates the possible shapes of triangle fuzzy numbers attached to categories (very low, low, very high). The numerical values are given by only one respondent who was selected for illustrative purposes.

Figure 2. Respondents' choices. Coding verbal categories with numerical values

Each respondent might define the categories 'vey high', 'high', etc. slighly differently. Using the numbers 1-100, please specify the number range for each category, marking the number from which each category starts, the number marking the middle of each category and the number closing each category.

	very low	low	medium	high	very high
beginning	0	10	...	...	80
middle	20	30	...	...	95
end	30	50	...	...	100



Source: author's work.

The main concern of a researcher using this measurement procedure is to guarantee a satisfactory level of adequacy, accuracy, and precision of the measurement. It is the researcher's responsibility to ensure that the variable used to measure attitudes and perceptions was described by an adequate measurement scale. The next issue is to identify the quantitative equivalents for verbal expressions used to mark various verbal categories of a natural language used as levels of the linguistic variable. When linguistic variables and verbal categories are coherent, it is possible to use the idea of Georg Rasch to enhance the uniformity in the interpretation of the assessments. Testing the adequacy of the measurement scale is vital to ensure that the measurement results are satisfactorily accurate.

3. An outline of the Item Response Theory. Model formulation

The idea of item analysis is based on the assumption that there is a hierarchy describing the quality of survey questionnaires. Discussing the issue, DePaoli et al. (2018, pp. 1299–1300) said: 'From a survey-development perspective, it is important to thoroughly examine the psychometric properties of any survey before finalizing the measure for broad use. […] there are other techniques based on the item response theory (IRT) framework that provide a more detailed assessment of the survey items'. In this context, placing the Rasch model on the broader outline of the Item Response Theory seems worthwhile (Royal et al., 2010; Zhu, 2002). In its mathematical concept, the Rasch model is a special case of the Item Response Theory, namely a one-parameter IRT model,[5] called the one-parameter linear model (1PL). In the literature on the subject, the IRT concept is compared with the Classical Test Theory (CTT). The extensive comparison of this kind may be found in the seminal work by R. Jabrailov et al. The authors state that 'The crucial difference between CTT and IRT is that in CTT the cutoffs are based on the distribution of the sum scores X, whereas in IRT they are based on the probability distribution' (Jabrayilov et al., 2016, p. 560). In the IRT context, the cutoff would be a certain quantile, usually a high percentile of the probability distribution in a functional population. Specialised websites are a comprehensive source of publications on the theory and applications of Rasch-type models (Jumailiyah, 2017).[6] An excellent comparison of the theoretical foundations of CTT and IRT may be found in Chapter 2 of the fundamental monography by DeMars (2018). The author discusses the accepted assumptions and formulates the specification of base model types and rules governing the process of designing the scale levels for items, along with practical issues, including reliability, required sample size, etc. The analytical review of cognitive and application aspects of CTT and IRT is presented by several authors, including Kong (2018) and Raykov and Marcoulides (2016). In their comprehensive analysis, Jabrayilov et al. (2016) showed the empirical outcome of the comparisons of the results of applications.

Due to readily available statistical packages, CTT are the easiest and most widely used techniques in the field of statistical analysis. The difference between IRT and classical analyses is that classical testing is usually performed on an entire set, whereas IRT more often concentrates on one item. What is more, the application of CTT is limited solely to the analysed population; sample and inference cannot be

[5] For simplicity, wherever possible and convenient, the Rasch model will be called the IRT model.

[6] Institute for Objective Measurement, https://www.rasch.org (accessed 20.11.2019). *Journal of Applied Measurement*, selected issues, http://jampress.org (accessed 20.11.2019). Rasch Measurement Transactions Contents, Archives of the Rasch Measurement, selected issues, https://www.rasch.org/rmt/contents.htm (accessed 20.11.2019).

extended onto items belonging to another sample,[7] although CCT can also generate item statistics. The question of the advantage of one approach over the other is subject to discussion. R. Jabrailov et al. state that 'The major advantage of CTT is its relatively weak theoretical assumptions, which make CTT easy to apply in many testing situations' (Jabrayilov et al., 2016, p. 559). Other authors claim that the application of CTT or IRT depends on the nature of testing situations, as each approach has its specific advantages and disadvantages. A full list of key advantages of IRT over CTT is given by Prieler (2007) (see Table 1).

Table 1. Key advantages of IRT over CTT for the analysis of change

CTT	IRT
The relation between the score and the ability level is based on the overall score across items.	A direct relationship is established between the ability level and the parameters of individual items (such as the difficulty of the item and discriminative power at different points in the distribution).
Emerging factors are seen as 'primary' influences on the test performance, with individual items being affected in different ways by other factors.	Emerging factors are less influenced by secondary factors, as much attention has been devoted to the issue of item homogeneity.
'Bad' items reduce the predictive power.	'Bad' items are eliminated.
Level of ability is defined in relation to a particular sample.	Level of ability can be defined independently of any sample.
Correlation is used to compare performance on repeated test occasions, which obscures the analysis.	No need to use correlation, so disadvantages are removed.
It is not possible to measure the significance of change at the individual level.	The significance of change at the individual level can be objectively measured.

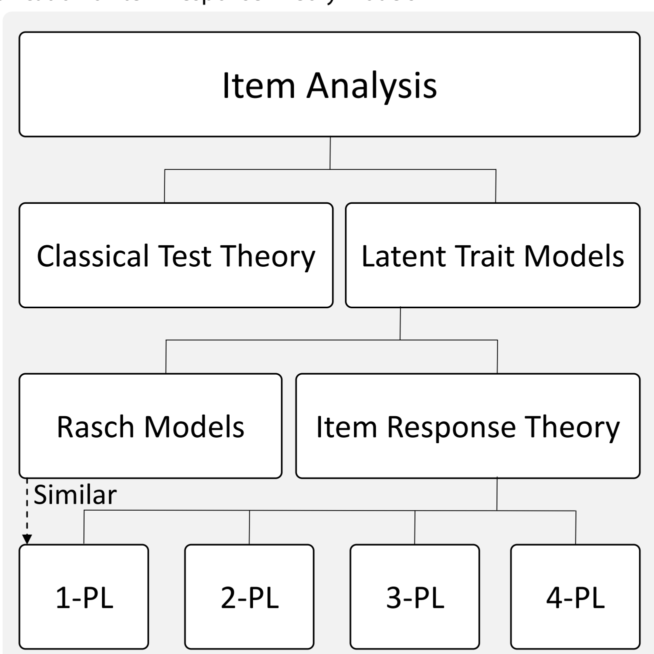
Source: author's work based on Prieler (2007, p. 701).

Jabrailov et al. (2016, p. 559) assert that researchers are able to see the possible advantages of using IRT over CTT. According to the authors, in the situation where tests consist of at least 20 items, the comparison of the CTT and IRT methods with regard to Type I error and detection rates showed that IRT is indeed superior to CTT in individual change detection. On the other hand, CTT appeared to be more effective at correctly detecting the change in individuals in shorter tests. Similar results were reported by Magno (2009). The popularity of the Item Response Theory results from social testing programs, conducted frequently and on a large scale, where IRT is referred to as modern psychometrics. This is a consequence of large-scale education assessment (e.g. PISA) or professional market testing (Edelen and Reeve, 2007). It could be said, in general terms, that IRT has many advantages over CTT that have brought it into more frequent use (Hambleton and Swaminathan, 1991c). It also seems that IRT has almost completely replaced CTT as a method of choice in some areas of application.

[7] Descriptive IRT vs. Prescriptive Rasch, https://www.rasch.org/rmt/rmt51f.htm (accessed 20.11.2019).

The question of the quality of measurement scales is connected with the issue of how adequate the survey questions were for respondents and to what extent they measured the ability of respondents to provide correct answers. The general framework of tests, questionnaires, and surveys makes it possible to reuse items such as e.g. questions or questionnaires, and thus they can appear repeatedly in several such structures. This is because their quality has already been verified, i.e. it is known how the questions are going to perform. In other words, IRT enables creating a reservoir of questions of foreseeable performance. Such a reservoir may constitute a kind of databank of questions and questionnaires (Combrinck, 2018; Yau and Yao, 2011; Linacre, 2002). Figure 3 provides an overview of the Item Response Theory with an indicated relation to the Classical Test Theory framework. The position of the Rasch model within the Item Response Theory models is also specified.

Figure 3. Classification of Item Response Theory models



Source: author's work based on Hambleton and Swaminathan (1991a).

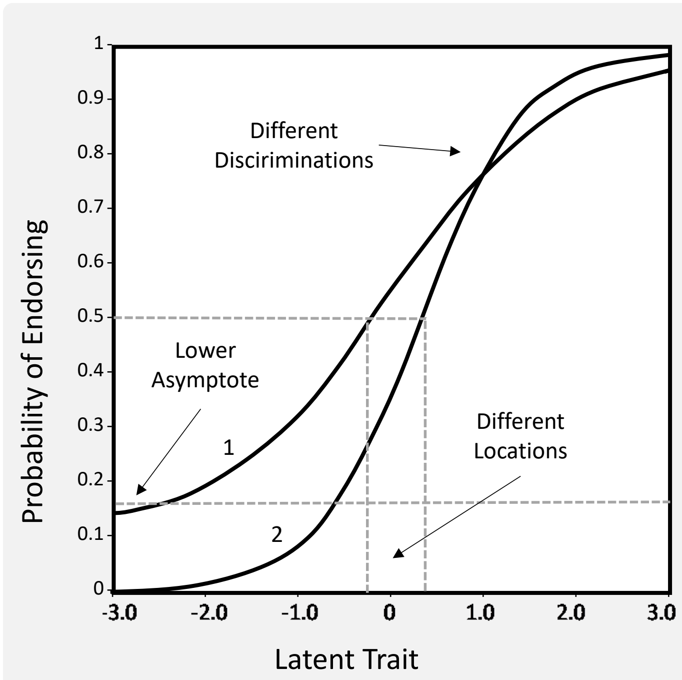
The Item Response Theory framework consists of three basic components:

- Item Response Function (IRF) – the function that relates the value of the latent characteristic (trait) to the probability of endorsing an item. The basis of the definition of the IRF is the concept of the latent variable defined as individual differences in reaction (assessment) to a construct (item). The IRF expresses the relation-

ship between a latent variable as defined above and the probability of endorsing an item. The concept of the IRF is used for modelling the relationship between the respondent trait level, the item properties, and the probability of endorsing the item;
- Item Information Function (IIF), which is considered as the indicator of the quality of an item, i.e. the item's ability to differentiate among respondents;
- invariance; provided that invariance is sustained, it is possible to estimate the item parameters from any position on the item response curve. Similarly, it is possible to estimate the parameters of an item from any group of respondents who have answered the item.

Item Characteristic Curves (ICC) is a very useful tool for representing results in a graphical form. ICC is created by the conversion of IRF into graphical functions which represent the respondent's abilities. The values of the ICC function represent the probabilities of endorsing an item by the respondent. The role of the item discrimination parameter is to illustrate the steepness of the IRF for each location of the item, i.e. the strength of the relation between the item and the value of the latent characteristic (Figure 4). Here, the analogy between the latent trait and the loadings in factor analysis might be observed. Items with a high discrimination parameter value may appear as ones which are better at differentiating respondents around the location point. In other words, minor changes in the latent trait lead to significant changes in the probability value. The latter statement also applies to the opposite – items with a low value of discrimination parameter a may be viewed as ones which are not as effective in differentiating respondents around the location point. Item location parameter b is defined as the amount of the latent trait which covers at least half of the probability of endorsing the item. The rule is that the higher the respondent's trait level while attempting to endorse the item, the higher value parameter b has. A similarity to the Classical Test Theory may be observed, as it involves the same complex task to determine Z scores. Additionally, as in the case of Z scores in CTT, usually the numeric values of parameter b range from −3 to +3. Applying parameter c, called item parameter guessing, increases the probability that respondents with a very low trait level may still choose the correct answer. One may expect that respondents presenting low trait levels, yet with a good intuition (and selecting answers at random) may still stand a chance of endorsing an item. It frequently occurs when multiple-choice testing is involved. It is expected that the parameter value should not vary considerably from the number of reciprocal choices. Parameter d of the IRF, called the item parameters upper asymptote, assumes that the probability that respondents with extremely high abilities will answer correctly is less than one. In other words, even such respondents are not always certain to make the correct choice.

Figure 4. Item Response Theory framework



Source: author's work based on Hambleton and Swaminathan (1991a).

The comprehensive four-parameter (4PL) logistic model may be denoted in the form of the following formula:

$$P(X = 1|\theta, a, b, c, d) = c + (d - c)\frac{e^{a(\theta-b)}}{1 + e^{a(\theta-b)}} \tag{1}$$

where:

θ – represents respondent trait level,

a – denotes the item's discrimination that determines the steepness of the IRF, alternatively called item parameters discrimination,

b – denotes the item difficulty that determines the location of the IRF, alternatively called item parameters location,

c – denotes a lower asymptote parameter for the IRF, alternatively called item parameters guessing; restricts the probability of endorsing the correct response for respondents with extremely low ability,

d – denotes an upper asymptote parameter for the IRF, alternatively called item parameters upper asymptote, which restricts the probability of endorsing the correct response for respondents with extremely high ability.

The left side of Equation (1) indicates the probability of responding correctly to a given item according to the key of answers. In calculations, the number of items and the number of respondents is denoted $i = 1, \dots , n, j = 1, \dots , N;$ respectively. The four-parameter logistic model (4PL) is the most complex and comprehensive logistic model of the Item Response Theory. A reformulation of Equation 1 to a simpler formula, the 3PL model, is possible by removing parameter d, and the 2PL and 1PL models could be obtained analogically – by removing parameters c and b, respectively.

As stated before, from the formal, mathematical point of view, the model proposed by Georg Rasch is identical to the basic Item Response Theory Model (1PL). What is different here is the approach of Rasch himself, who believed the model to be superior to data. Following his way of thinking, if some data does not fit the model, it should be discarded. Additionally, Rasch's specification does not allow the estimation of abilities for extreme items and persons. In principle, the Rasch model is designed for categorical data. The model's elegant mathematical form renders it suitable for extensions that allow greater flexibility in handling complex samples relating to collections of tasks representing different domains. Extensions of the Rasch model are enhanced by additional structural elements that account for differences among diverse populations or observed variables.

4. The assumptions of the Item Response Theory models

The Item Response Theory is a universal paradigm. The variants of models are designed to suit the specific qualities of any given population. A set of common assumptions constitutes the base for the specification, the assessment of applicability and the rules for the interpretation of results (NAP, 2017; Tinsley and Brown, 2000; Hambleton and Swaminathan, 1991b).

The Invariance assumption. Invariance is the position of the latent trait which may be estimated by any item with a known Item Response Function. Invariance means here that the item characteristics are independent of the population to which they are applied. The statement concerns the linear transformation of items. So, invariance means that regardless of which questions are being asked, the assessment of the level of respondent's abilities remains the same. In other words, the assessment of the level of respondents' abilities does not change when the questions do. On the other hand, item parameters are not determined by a particular group within the sample of respondents or inside their linear transformation. The property or assumption of invariance is crucial for socio-economic measurement. It makes it possible to:

- link scales that measure the same construct;
- implement computerised adaptive testing;
- compare respondents, also when they answered different items on the scale.

Unidimensionality assumption. It is assumed that in the conventional Item Response Theory models, considered to be one-dimensional, parameter theta characterises individual differences. As a consequence, the item covariance in the discussed model specification includes a single common factor, i.e. a latent trait or a latent feature, which is estimated by means of specialised factor analytic models for dichotomous items (Maydeu-Olivares et al., 2011; Kappenburg -ten Holt, 2014). There are also multidimensional IRT models, but they are not commonly used in applied research (Immekus et al., 2019; Hartig and Hoehler, 2009; Ackerman, 2005).

Local Independence Assumption. The Local Independence (LI) assumption indicates that item responses are uncorrelated, provided that control over the latent trait is established. The LI and unidimensionality are naturally related. The former is liable to violations which are called local dependencies. Even highly univocal scales can be susceptible to violations of local independence, which may occur, for example, due to item content dependence. Local Independence Assumption violations may lead to serious consequences (local dependencies), such as the following:

- they may distort values of item parameter estimates, which in practice means that item slopes are steeper than they really are;
- they may cause the scale to look more precise than it actually is;
- the occurrence of Local Dependence (LD) may lead to a false conclusion about the invalidity of the scale, which may essentially define or dominate the latent trait in a construct where a strong correlation between two or more items exists.

Therefore, the violations of local independence have to be addressed. H. Wainer and G. Kiely recommended forming testlets by combining locally dependent items for this purpose. A testlet is defined as an aggregation of items which are based on a single stimulus, such as, for example, a reading comprehension test. In this case, a testlet is a passage and the set of four to twelve items that accompany it (Wainer and Kiely, 1987; Sireci et al., 2005). Alternatively, LD may be addressed by removing one or more of the LD items from the scale in order to achieve local independence.

Qualitatively homogeneous population assumption. The key assumption of the Item Response Theory models states that the same IRF applies to all members of the respondent's population. The violation of the qualitatively homogeneous population assumption, called differential item functioning, means that a violation of the invariance property occurred. If an item has a different IRF for two or more groups, it may lead to false conclusions, e.g. for respondents who are equal in terms of the latent feature, different probabilities of the expected scores of endorsing an item could be estimated.

Monotonicity assumption. When specifying logistic IRT models, it is assumed that as the trait level increases, so does the probability of endorsing an item. In mathematical terms, models have the form of a monotonically increasing function. As a consequence, in the situation where this assumption is violated, applying the logistic form of the model to describe item response data becomes pointless.

5. Item Response Theory model – application

The concept of the Test Response Curve (TRC) is crucial for the interpretation of the results from the point of view of their applicability. Since Item Response Functions are additive, the researcher can combine items to create a Test Response Curve. TRC describes the latent trait's dependency on the number of considered items. An equally important analytical tool is the Item Information Function (IIF), where the item reliability is replaced by the item information. Each IRF can be transformed into the IIF. The values of IIF provide a precise representation of an item at each level of the latent trait. The information has the form of an index representing the item's ability to differentiate among individuals. The standard error of measurement, which is a variance of the latent trait level, may be interpreted in such a way that more information means less error, and vice versa. According to the standard error definition, the measurement error is expressed on the same metric scale as the latent trait level, so it can be used to build confidence intervals. The Item Information Function is crucial in the process of creating a quality description of the measurement scale. It is possible to extract several of its characteristics, including the following:

- a difficulty parameter, understood as the location of the highest information point;
- a discrimination parameter, understood as the height of the information;
- large discriminations, i.e. the high and narrow IIFs; a high level of precision is expressed by a narrow range;
- low discrimination, i.e. short and wide IIFs; a low level of precision is expressed by a broad range.

In the one-parameter logistic model, the discrimination parameter is fixed for all items, and, accordingly, all the Item Characteristic Curves corresponding to different items on the measurement scale are parallel along the ability scale.

The Item Information Function values are the measurements of the amount of information provided by individual items. Those values may be calculated by multiplying the probability of endorsing a correct response by the probability of endorsing an incorrect answer.

Since the Item Information Functions are additive, the aggregate function may be understood as the Test Information Function (TIF). The TIF may be used for the assessment of the test as a whole, and in particular to identify parts of the characteristic range that are most precise and perform best. In Polish literature, the papers by Jefmański (2014) and Brzezińska (2016) constitute the first attempts to show the potential of this methodology.

6. Assessment of scale adequacy

The data which was used for the exercise that served as an example of the assessment of the scale adequacy within the Rasch theoretical framework was collected at the Wroclaw University of Economics, i.e. the university the author works at. The data was collected according to the procedure described in part three of this article and by means of the Computer Assisted Personal Interview survey (Lynn, 2019). Respondents were asked to specify their opinions concerning a set of innovative products (smartphones), using computer screens illustrated in Figures 1 and 2. The products were described by five characteristics and the overall assessment variable. Altogether, the sample consisted of over 450 sets of assessments submitted by the respondents. Since the group of respondents was selected using the convenience approach, the study should then be considered a pilot study whose aim was to verify the possibility of applying the proposed approach. The questionnaire covered the following issues: respondents' preferences as to the leading smartphone brands, the available smartphone applications, and the key characteristics of the devices. The measurement results were collected in the form of unconventional fuzzy numbers. Figure 5 illustrates the frequency (%) of chosen answers defining the beginning, middle, and upper limit of fuzzy numbers corresponding to the individual categories of the verbal grades.

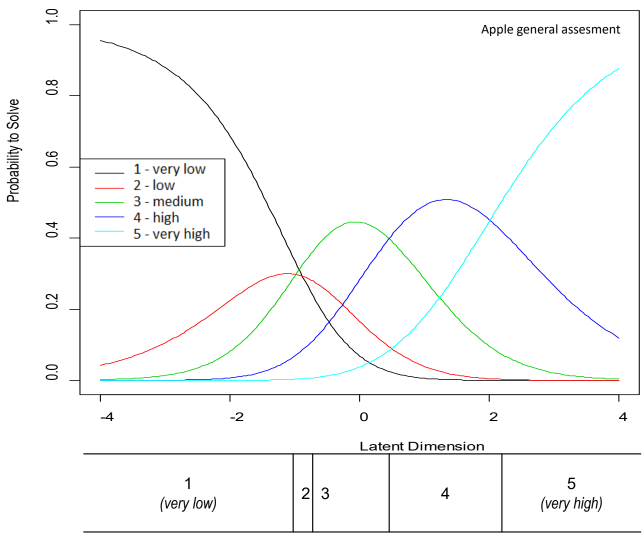
Figure 5. Frequency (in %) of answers defining the beginning, middle, and upper limit of fuzzy numbers corresponding to individual categories of verbal grades

Frequency	Very low		Low			Medium			High			Very high	
%	beginning middle	end	beginning	middle	end	beginning	middle	end	beginning	middle	end	beginning	middle end
0	100	2	0	0	0	0	0	0	0	0	0	0	0
10	0	13	22	0	0	0	0	0	0	0	0	0	0
20	0	54	57	16	2	6	0	0	1	0	0	0	0
30	0	20	16	56	17	18	1	0	3	0	0	0	0
40	0	7	4	21	46	47	12	2	2	1	0	0	0
50	0	3	1	5	25	24	55	9	9	3	0	1	0
60	0	1	0	0	2	3	11	6	5	1	0	1	0
70	0	0	0	1	4	2	18	48	47	13	2	2	0
80	0	0	0	0	2	0	3	26	28	49	9	8	0
90	0	0	0	0	2	0	0	5	5	25	50	49	0
100	0	0	0	0	1	0	0	3	0	8	39	39	100

Source: author's work.

Respondents defined numerical values of the verbal categories with various forms of triangular fuzzy numbers. Some respondents attempted not to overlap, some tried to keep the equal length, while some others to cover the full range of possible values (from 0 to 100). A very useful tool for graphical representation of results is the Item Characteristic Curves. The ICC is the result of the IRF conversion which takes the form of graphical functions representing the respondent's ability.

Figure 6. Characteristic curves for subscale items (Item Characteristic Curves)



Source: author's work in R (eRm package).

The values of the ICC function are the probabilities of endorsing the item by the respondent. The general assessment of Apple smartphone is shown in Figure 6 to illustrate the location of borders between categories. Respondents are very far from a uniform distribution in their statements; the widest range is attributed to categories very low and very high, while the category low has a very narrow range attributed by respondents, which came as a surprise. On the other hand, there were some respondents who defined very narrow ranges of codes for their verbal categories, but also approximately a third of them coded their categories with wide and frequently overlapping ranges. Subsequent illustrations for the remaining characteristics shown in Figure 7 confirm that it was worthwhile to allow unconventional fuzziness for characteristics assessment. The uneven lengths of numerical codes attached to individual verbal categories prove that respondents attach various meanings and diverse, hidden and latent values connected to their subjective perception of product features. The interpretation of the assessments is strongly related to the shape of the resulting triangle. Narrow triangles signify strong opinions anchored in a well-established view resulting in the knowledge of the rules for interpreting attribute values. Wide triangles, on the other hand, signify the lack of strong opinions, no solid views and the lack of determination in formulating opinions.

In addition, it can be inferred that these respondents do not have proper knowledge about the products studied, or their general knowledge is poor. This leads to a specific interpretation of product features, i.e. the subjective perception of the product properties. As a result, formulated assessments of the subjective perception of individual product properties take the form of triangles with very broad foundations.

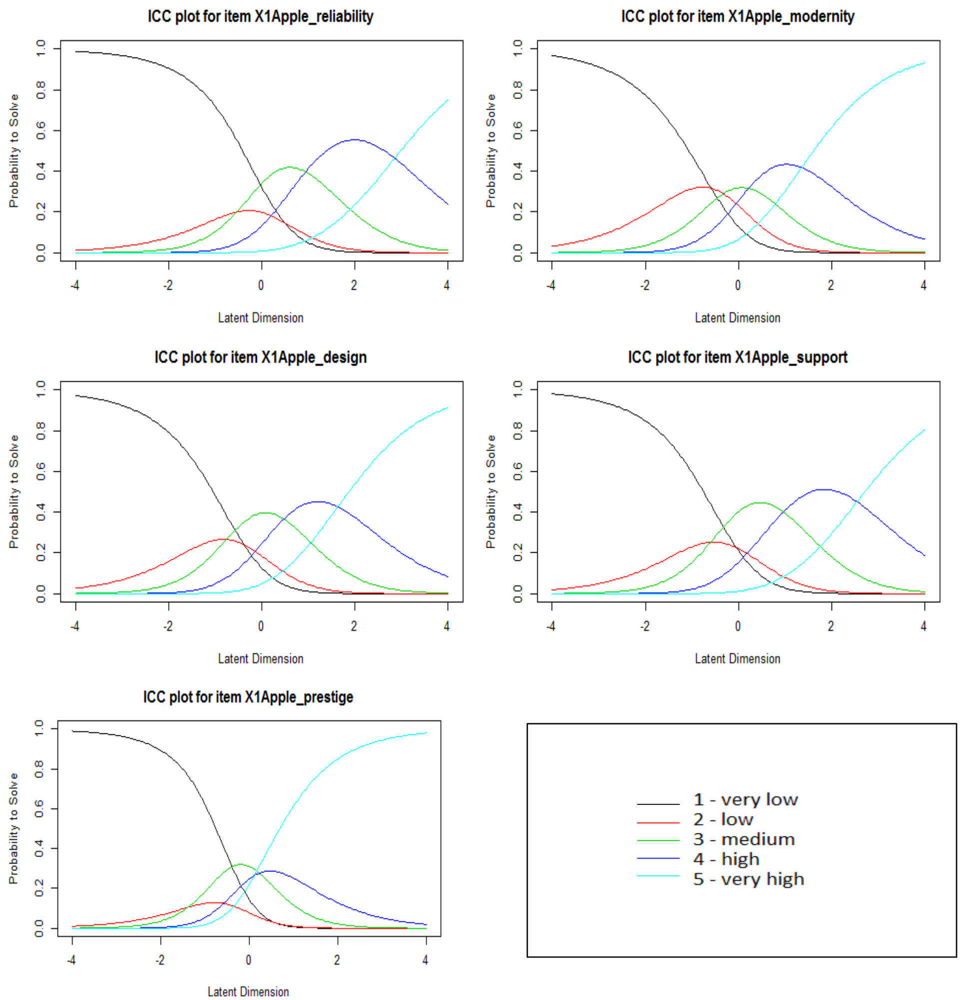
A response to the latter observation is the probabilistic test theory, which examines the probability which may be attached to the respondents' possible answers to a given scale item. Scale items, called statements, are a function of a hidden variable that specifies the level of the respondent's ability to measure the socio-economic phenomenon properly, understood as the ability to give a true answer to the scale item. On the other hand, the level of difficulty of test statements should be assessed. Both tasks may be done using the Rasch specifications.

The Item Information Function curve indicates the quality of an item, i.e. the item's ability to differentiate among respondents (Figure 8). The definition of the Response Function is based on a concept of the latent variable defined as individual differences in reaction, manifested in the assessment of a construct, sometimes referred to as an item. The IRF characterises the relation between such latent variable and the probability of endorsing an item.

As was mentioned before, Item Information Functions are additive and the aggregate function is called Test Information Function. As shown in Figure 9, TIF may be used as a complete assessment of a test. It is also helpful while identifying parts of the characteristic range that are most precise and perform best. Additionally, it might be considered an indicator of item quality, i.e. the item's ability to differentiate among respondents. Linguistic expressions may be coded as fuzzy triangular numbers by means of a partial credit model framework. This kind of model belongs to the family of models used for the theory of response to scale items. The ranges of the domains may be determined on the basis of the intersection points of the characteristic curves of adjacent categories (Linacre, 2000, 2002). The most recent results may be found in the summary provided on a specialised website.[8]

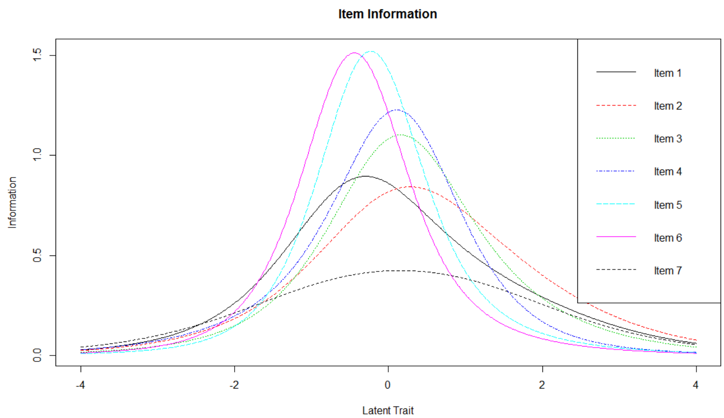
[8] Model selection: Rating Scale Model (RSM) or Partial Credit Model (PCM), https://www.rasch.org/rmt/rmt1231.htm (accessed 20.11.2019).

Figure 7. Characteristic curves for subscale items used for Apple assessment



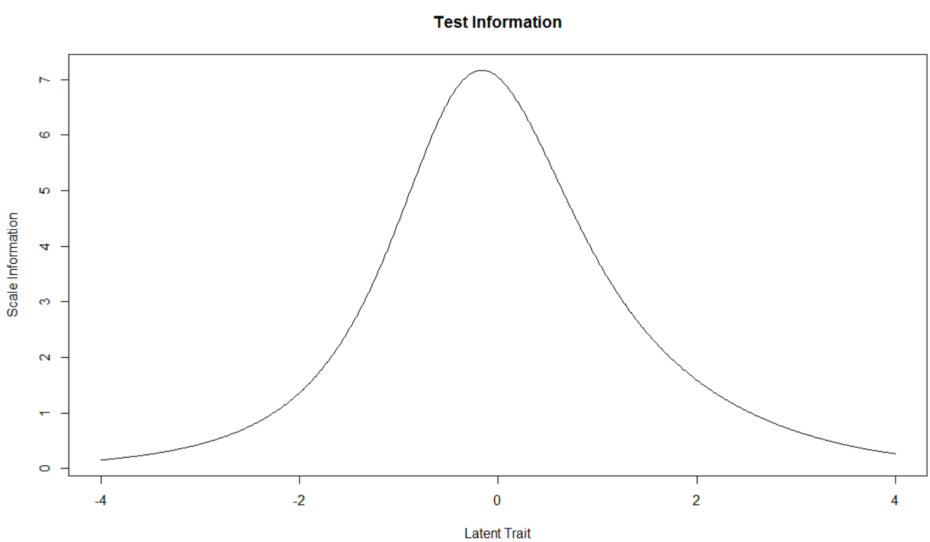
Source: author's work in R (eRm package).

Figure 8. Item Information Curves



Source: author's work in R (eRm package).

Figure 9. Aggregate function understood as Test Information Function



Source: author's work in R (eRm package).

Table 2 introduces formulas that enable the establishment of the parameters of triangular fuzzy numbers for each of the categories outlined within the i-th item. The example presents a rating scale with five verbal categories on the ordinal scale: very low (VL), low (L), medium (M), high (H), and very high (VH).

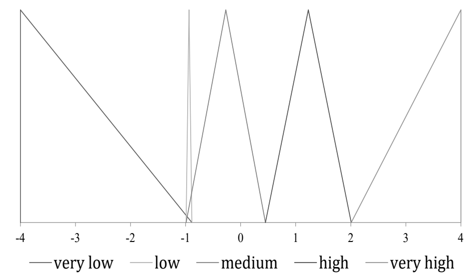
Table 2. Determination of the values of parameters of triangular fuzzy numbers for verbal categories

Parameters of fuzzy numbers	Categories				
	1 – Very low	2	3	4	5 – Very high
α_1	-4	τ_{i1}	τ_{i2}	τ_{i3}	τ_{i4}
α_2	-4	$\frac{\tau_{i1}+\tau_{i2}}{2}$	$\frac{\tau_{i2}+\tau_{i3}}{2}$	$\frac{\tau_{i3}+\tau_{i4}}{2}$	4
α_3	τ_{i1}	τ_{i2}	τ_{i3}	τ_{i4}	4

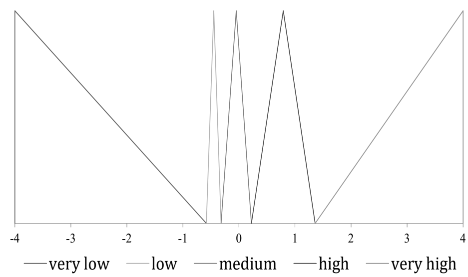
Source: author's work based on Model selection: Rating Scale Model (RSM) or Partial Credit Model (PCM), https://www.rasch.org/rmt/rmt1231.htm (accessed 20.11.2019); Linacre (2000, 2002).

It should be stressed that the category represented by the term *low* is hardly ever chosen, which seen from the methodical point of view might lead to the conclusion that this category could be removed from the scale, as according to the respondents' choices, it hardly describes the considered phenomenon at all. On the other hand, however, it might be that most respondents evaluate the product characteristic positively, whereas a negative value is chosen only by those respondents who do not accept the brand at all (for example see Figure 10).

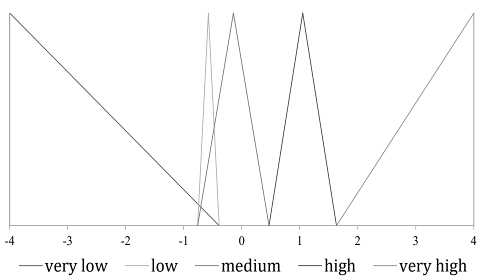
Figure 10. A graphical form of triangular fuzzy numbers assigned to verbal categories for the Apple brand



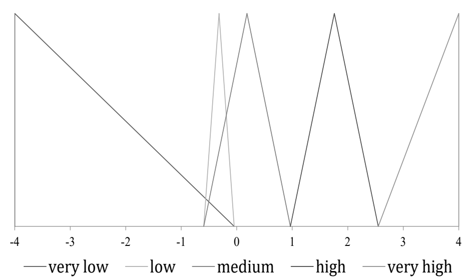
10.1. Apple – reliability



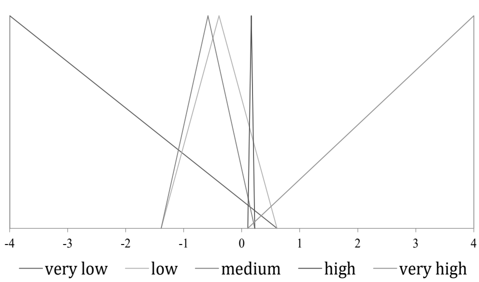
10.2. Apple – modernity



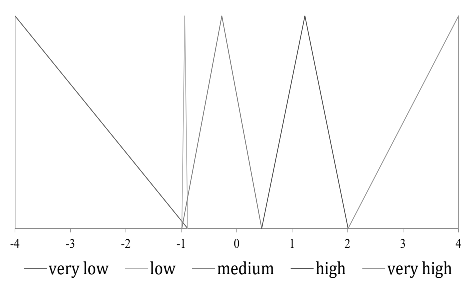
10.3. Apple – design



10.4. Apple – support



10.5. Apple – prestige



10.6. Apple – general assessment

Source: author's work.

7. Concluding remarks

In conclusion, it may be stated that the measurement of socio-economic phenomena, including the subjective perception of the material wellbeing of households, the quality of durable goods in households, and the assessment of the quality of goods available to the members of the household requires special tools. It has been proven that one of the most useful and powerful of such tools is a linguistic scale. Specialised procedures are necessary to code verbal terms with numerical equivalents. The author suggests the use of unconventional fuzzy numbers for this purpose.

The new proposal on how to perform the assessment and measurement of the scale adequacy proved to be useful and effective. The idea of the discussed assessment technique becomes relevant when the measurement results of a linguistic scale are coded with numerical equivalents. The author is interested in increasing the objectivity of the results of the measurement of households' subjective wellbeing as well as the subjective perception of households' endowment with durables. The process of testing the author's theory included the measurement of the subjective perception of the socio-economic phenomena on a linguistic scale. The respondents coded their own subjective perceptions with fuzzy numbers, usually with unconventional forms of fuzziness. This confirmed the author's supposition of the diversity of individual perceptions and assessment of phenomena. The core of the author's interest is focused on the use of unconventional fuzzy numbers.

As Figure 6 indicates, it is possible to establish numerical delimitation points between verbal categories. The technique proves useful in the design of survey questionnaires. The framework for the assessment of scale adequacy is provided by the Item Response Theory. The author tested the usefulness of the one-parameter variant of the ITR, often called the Rasch model. That study demonstrated that the interpretation of assessments can be strongly related to the shape of the resulting triangles. Hence, it is advisable, and sometimes necessary, to analyse the values given by those respondents who do not have strong, well-grounded opinions (which is illustrated by wide ranges of fuzzy numbers). Respondents with such a manner of assessment represent a completely different perception of subjective values of verbal categories. Similarly, those respondents who have strong, well-grounded opinions, which are manifested in narrow ranges of fuzzy numbers, need a different approach in the interpretation of measurement results. They demonstrate a wider knowledge and are more focused on the differentiation between latent values behind verbal categories.

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Multilane turnpike in the non-stationary input-output economy with von Neumann temporary equilibrium

Emil Panek[a]

Abstract. In the author's earlier papers concerning asymptotic characteristics of the optimal growth processes in non-stationary Gale economies with multilane production turnpikes, it is assumed that production technology used in time period t may also be used in the next period. Such an assumption, relevant for short periods, is difficult to justify in the longer term.

The paper contains a proof of the so called 'weak' effect of the multilane turnpike in a non-stationary Gale economy with changing technology, where this assumption has been suspended.

Keywords: non-stationary Gale economy, von Neumann temporary equilibrium, multilane turnpike

JEL: C62, C67, O41, O49

1. Basic assumptions and definitions

We assume that in the economy there are n used and/or produced goods, where n is an integer and is finite. It is assumed that the time t is discrete, $t = 0, 1, 2, \ldots$ We let $x(t) = \left(x_1(t), \ldots, x_n(t)\right) \geqq 0$ denote goods that are used in period t (an input vector or expenditure vector) and $y(t) = \left(y_1(t), \ldots, y_n(t)\right) \geqq 0$ goods that are produced in this period from inputs $x(t)$ (an output vector or production vector).[1] We say that $\left(x(t), y(t)\right)$ represents / describes a feasible production process (due to disposable technology in the economy in period t). We let $Z(t)$ denote the set of all technologically feasible production processes in period t and we call it Gale's production space (technological set) in period t. The notation $(x, y) \in Z(t)$ (or equivalently $\left(x(t), y(t)\right) \in Z(t)\big)$ means that the economy in period t can produce the output $y(t)$ using inputs $x(t)$. Production spaces $Z(t), t = 0,1, \ldots$ are nonempty subsets of $\mathbb{R}^{2n}_+$, satisfying the following conditions:

(G1) $$\forall (x^1, y^1) \in Z(t)\ \forall (x^2, y^2) \in Z(t)\ \forall\, \lambda_1, \lambda_2 \geq \\ \geq 0\ \left(\lambda_1 (x^1, y^1) + \lambda_2 (x^2, y^2) \in Z(t)\right)$$

(homogeneity and additivity of production processes),

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[1] If $a \in R^n$, then $a \geqq 0$ means that $\forall i (a_i \geq 0)$, in contrast to $a \geq 0$, which means $(a \geqq 0 \wedge\ a \neq 0)$.

(G2) $$\forall (x,y) \in Z(t)\ (x = 0 \Rightarrow y = 0)$$

(no land-of-Cockaigne condition),

(G3) $$\forall (x,y) \in Z(t)\ \forall x' \geqq x\ \ \forall\, 0 \leqq y' \leqq y\ \big((x',y') \in Z(t)\big)$$

(costless waste condition),

(G4) Production spaces $Z(t)$ are closed subsets of $\mathbb{R}^{2n}_{+}$.

From **(G1), (G2), (G4)** it follows that each production space $Z(t)$ is a closed convex cone in $\mathbb{R}^{2n}_{+}$ with a vertex at 0 and the property that if $(x,y) \in Z(t)$ and $(x,y) \neq 0$, then $x \neq 0$. We are only interested in such non-trivial processes $(x,y) \in Z(t)\backslash\{0\}$. These conditions imply that:

$$\forall t\ \forall (x,y) \in Z(t)\backslash\{0\}\ \exists \alpha(x,y) = \max\{\alpha|\ \alpha x \leqq y\} = \min_i \frac{y_i}{x_i} \geq 0$$

A non-negative number $\alpha(x,y)$ is called the technological efficiency rate of the process (x,y). The function $\alpha(\cdot)$ is positively homogenous of degree zero on $Z(t)\backslash\{0\}$ and

$$\exists \big(\bar{x}(t),\bar{y}(t)\big) \in\ Z(t)\backslash\{0\}\ \left(\alpha\big(\bar{x}(t),\bar{y}(t)\big) = \max_{(x,y)\in Z(t)\backslash\{0\}} \alpha(x,y) = \alpha_{M,t}\right) \geq 0$$

see e.g. Takayama (1985, Th. 6.A.1; after replacing $Z(t)$ with Z and $\alpha_{M,t}$ with α_M). The process $\big(\bar{x}(t),\bar{y}(t)\big)$ is called an optimal production process while $\alpha_{M,t}$ an optimal efficiency rate in the non-stationary Gale economy in period t. Due to the positive homogeneity of degree zero of the function $\alpha(\cdot)$, an optimal production process multiplied by any positive number is also an optimal production process

$$\forall \lambda > 0 \left(\ \alpha\big(\bar{x}(t),\bar{y}(t)\big) =\ \ \alpha\big(\lambda\bar{x}(t),\lambda\bar{y}(t)\big)\right)$$

Let us assume that:

(G5) $$\forall t\ \forall i \in \{1,2,\dots,n\}\ \exists (x,y) \in Z(t)\big((y_i) > 0\big)$$

(the economy has a technology that at any period t allows the production of each good). This assumption along with **(G1)** ensures that the optimal efficiency rate $\alpha_{M,t}$ is always (at any period t) positive. Let us denote:

$$Z_{opt}(t) = \{(\bar{x}, \bar{y}) \in Z(t) \backslash \{0\} | \alpha(\bar{x}, \bar{y}) = \alpha_{M,t} > 0\}$$

This set consists of all the optimal production processes in a non-stationary Gale economy in period t. All sets $Z_{opt}(t) \subset Z(t), t = 0,1, \dots$ are cones that are contained in $\mathbb{R}^{2n}_{+}$ (without 0).[2] From **(G3)** it follows that if $(\bar{x}, \bar{y}) \in Z_{opt}(t)$, then also $(\bar{x}, \alpha_{M,t}\bar{x}) \in Z_{opt}(t)$ and $(\bar{y}, \alpha_{M,t}\bar{y}) \in Z_{opt}(t)$. The vector $s(t) = \frac{\bar{y}(t)}{\|\bar{y}(t)\|}$ represents the production structure in the optimal process $(\bar{x}(t), \bar{y}(t)) = (\bar{x}, \bar{y}) \in Z(t) \backslash \{0\}$[3] Assuming **(G1)–(G5),** the sets[4]

$$S(t) = \left\{s(t) | \exists (\bar{x}(t), \bar{y}(t)) \in Z_{opt}(t) \left(s(t) = \frac{\bar{y}(t)}{\|\bar{y}(t)\|}\right)\right\}, t = 0, 1, \dots$$

of the production structure vectors in all optimal processes in each period t are nonempty, compact and convex; Panek (2016, Th. 2). If $s = s(t) \in S(t)$, then the ray

$$N_s^t = \{\lambda s | \lambda > 0\} \subset \mathbb{R}^n_+$$

is called a single production turnpike (von Neumann's ray) in a non-stationary Gale economy in period t. The set

$$\mathbb{N}^t = \bigcup_{s \in S(t)} N_s^t = \{\lambda s | \lambda > 0, s \in S(t)\}$$

of all the singular turnpikes N_s^t forms a multilane production turnpike that is accessible in a non-stationary Gale model in period t. Each multilane turnpike $\mathbb{N}^t, t = 0,1, \dots$, is a cone in $\mathbb{R}^n_+$ not containing 0.

In Panek (2018, Lemma 1) we prove that if in a non-stationary Gale economy satisfying conditions **(G1)–(G5)** in a certain period t inputs structure $\frac{x}{\|x\|}$ or outputs structure $\frac{y}{\|y\|}$ in a process $(x, y) \in Z(t) \backslash \{0\}$ is different than a turnpike's structure, then its technological efficiency is lower than optimal:

---

[2] See Panek (2016, Th. 1; after replacing $Z(t), \alpha_{M,t}$ with α_M).

[3] Here and on, if $a \in \mathbb{R}^n$, then $\|a\| = \sum_{i=1}^n |a_i|, \frac{a}{\|a\|} = \left(\frac{a_1}{\|a\|}, \frac{a_2}{\|a\|}, \dots, \frac{a_n}{\|a\|}\right)$

[4] Equivalently $S(t) = \left\{s(t) | \exists (\bar{x}(t), \bar{y}(t)) \in Z_{opt}(t) \left(s(t) = \frac{\bar{x}(t)}{\|\bar{x}(t)\|}\right)\right\}$.

$$\forall t\ \forall (x,y) \in Z(t)\backslash\{0\}\left(\frac{x}{\|x\|} \notin S(t) \vee \frac{y}{\|y\|} \notin S(t) \Rightarrow \alpha(x,y) < \alpha_{M,t}\right) \tag{1}$$

This condition is important in the proofs of turnpike theorems (Theorems 3 and 4).

Let $p(t) = \big(p_1(t), \dots ,\ p_n(t)\big) \geq 0$ denote the price vector in an economy in period t. If $\big(x(t), y(t)\big) \in Z(t)\backslash\{0\}$, then $\langle p(t), x(t)\rangle = \sum_{i=1}^{n} p_i(t)x_i(t)$ represents the value of expenditures incurred in that economy in period t (expressed in prices $p(t)$), while $\langle p(t), y(t)\rangle = \sum_{i=1}^{n} p_i(t)y_i(t)$ is the value of outputs produced in period t from inputs $x(t)$. Let

$$\beta\big(x(t), y(t), p(t)\big) = \frac{\langle p(t), y(t)\rangle}{\langle p(t), x(t)\rangle} \geq 0$$

($\langle p(t), x(t)\rangle \neq 0$) denote the economic efficiency rate of the process $\big(x(t), y(t)\big)$ in period t (with prices $p(t)$). Let us take any optimal process $\big(\bar{x}(t), \bar{y}(t)\big) \in Z_{opt}(t)$. Then

$$\alpha_{M,t}\bar{x}(t) \leqq \bar{y}(t) \tag{2}$$

□ **Theorem 1.** Assuming **(G1)–(G5)**:

$$\forall t \geq 0\ \exists \bar{p}(t) \geq 0\ \ \forall (x,y) \in Z(t)\big(\langle \bar{p}(t), y\rangle \leq \alpha_{M,t}\langle \bar{p}(t), x\rangle\big) \tag{3}$$

Proof.[5] According to **(G1)** for any t we have $(0,0) \in Z(t)$ and hence, regarding **(G3)**,

$$\forall t\ \forall i \in \{1, 2, \dots, n\}\left(\left(e^i, 0\right) \in Z(t)\right)$$

where $e^i = (0, \dots, 1, \dots, 0) \in \mathbb{R}^n$ is an n-dimmensional vector with 1 as the i-th coordinate.

Let us choose any period t and let us define the set

$$C(t) = \left\{c \mid \forall (x,y) \in Z(t)\ \left(c = \alpha_{M,t}x - y\right)\right\}$$

[5] Proof based on Panek (2019b, Th. 1).

This set is a convex cone in $\mathbb{R}^n$ (as a linear image of a cone $Z(t)$), which does not contain negative vectors. Indeed, if $C(t)$ contained a negative vector c', then there would exist a production process $(x', y') \in Z(t)$, such that $c' = \alpha_{M,t}x' - y' < 0$, and therefore $\alpha_{M,t}x' < y'$. This would mean that

$$\exists \varepsilon' > 0 \left(\alpha_{M,t} = \max_{(x,y) \in Z(t) \setminus \{0\}} \alpha(x, y) \geq \alpha(x', y') \geq \alpha_{M,t} + \varepsilon' \right)$$

and this contradicts the definition of the optimal efficiency rate $\alpha_{M,t}$.

Since $\left(e^i, 0\right), i = 1,2, \dots, n$ belongs to $Z(t)$, therefore vectors

$$c^i = \alpha_{M,t}e^i - 0 = \left(0, \dots, \alpha_{M,t}, \dots, 0\right), \qquad i = 1, 2, \dots, n$$

belong to $C(t)$ (with $\alpha_{M,t} > 0$ on the ith coordinate). Then, the hyperplane separation theorem implies:

$$\exists \bar{p}(t) \neq 0 \; \forall c \in C(t)(\langle \bar{p}(t), c \rangle \geq 0)$$

In particular,

$$\langle \bar{p}(t), c^i \rangle = \alpha_{M,t}\bar{p}_i(t) \geq 0, \qquad i = 1, 2, \dots, n$$

so $\bar{p}(t) \geq 0$. Hence, condition (3) follows from $c = \alpha_{M,t}x - y$. ■

2. Temporary von Neumann equilibrium

It follows from (2) and (3) that

$$\forall \left(\bar{x}(t), \bar{y}(t)\right) \in Z_{opt}(t)\left(\langle \bar{p}(t), \bar{y}(t) \rangle = \alpha_{M,t} \langle \bar{p}(t), \bar{x}(t) \rangle \geq 0 \right)$$

which means that it is theoretically possible that $\langle \bar{p}(t), \bar{y}(t) \rangle = 0$ (zero production value in the optimal process $\left(\bar{x}(t), \bar{y}(t)\right)$. This unrealistic case does not occur when the following condition is met in the economy:

(G6)
$$\forall t \geq 0 \; \forall (x, y) \in Z(t) \setminus \\ \setminus \{0\} \left(\alpha(x, y) < \alpha_{M,t} \Rightarrow 0 \leq \beta\left(x, y, \bar{p}(t)\right) = \frac{\langle \bar{p}(t), y \rangle}{\langle \bar{p}(t), x \rangle} < \alpha_{M,t} \right)$$

(no production process that does not have the highest technological efficiency can achieve the maximum economic efficiency).

□ **Theorem 2.** If $\bar{p}(t)$ are prices from Theorem 1, and conditions **(G1)–(G6)** are satisfied, then

$$\forall t\ \forall\big(\bar{x}(t), \bar{y}(t)\big) \in Z_{opt}(t) \subset Z(t)\ (\bar{p}(t), \bar{y}(t) = \\ = \alpha_{M,t}\langle \bar{p}(t), \bar{x}(t)\rangle > 0) \tag{4}$$

Proof is similar to the proof of Theorem 1 in Panek (2018) (after substituting $\big(\bar{x}(t), \bar{y}(t)\big) \in Z_{opt}(t)$, $\alpha_{M,t}$ instead of $(\bar{x}, \bar{y}) \in Z_{opt}$, α_M). ■

When the optimal efficiency rate $\alpha_{M,t}$, optimal production process $\big(\bar{x}(t), \bar{y}(t)\big) \in$ $\in Z_{opt}(t) \subset Z(t)$ and prices $\bar{p}(t) \geq 0$ meet the following conditions:

$$\alpha_{M,t}\bar{x}(t) \leqq \bar{y}(t) \tag{5}$$

$$\forall (x, y) \in Z(t)\big(\langle \bar{p}(t), y\rangle \leq \alpha_{M,t}\langle \bar{p}(t), x\rangle\big) \tag{6}$$

$$\langle \bar{p}(t), \bar{y}(t)\rangle > 0 \tag{7}$$

we then say that they define the state of the temporary von Neumann equilibrium in a non-stationary Gale economy in period t. The vector $\bar{p}(t)$ is called the equilibrium price vector in period t. It follows from conditions (5)–(7) that in the equilibrium:

$$\beta\big(\bar{x}(t), \bar{y}(t), \bar{p}(t)\big) = \frac{\langle \bar{p}(t), \bar{y}(t)\rangle}{\langle \bar{p}(t), \bar{x}(t)\rangle} = \max_{(x,y)\in Z(t)\setminus\{0\}} \beta\big(x, y, \bar{p}(t)\big) = \\ = \alpha\big(\bar{x}(t), \bar{y}(t)\big) = \alpha_{M,t} > 0$$

i.e. in the temporary equilibrium in any period t, the economic efficiency of production equates with the technological efficiency, and in each case this is the highest possible efficiency for the economy. In the non-stationary Gale economy, under conditions **(G1)–(G6)**, the temporary equilibrium states always exist (in all periods of time). They are formed in each period t by a triple $\{\alpha_{M,t}, \big(\bar{x}(t), \bar{y}(t)\big),\ \bar{p}(t)\}$ with any optimal production process $\big(\bar{x}(t), \bar{y}(t)\big) \in Z_{opt}(t)$. Prices vector $\bar{p}(t)$ and production process $\big(\bar{x}(t), \bar{y}(t)\big)$ in the temporary equilibrium are defined up to the structure (i.e. any positive multiple of a temporary equilibrium price vector is also an equilibrium price vector and any positive multiple of an optimal process is also optimal process).

Let us consider any production process $(x, y) \in Z(t)\setminus\{0\}$ and let

$$d(x, \mathbb{N}^t) = \inf_{x'\in\mathbb{N}^t} \left\| \frac{x}{\|x\|} - \frac{x'}{\|x'\|} \right\| \tag{8}$$

be a measure of the distance of the vector x from the multilane turnpike $\mathbb{N}^t$ in period t. If conditions **(G1)–(G6)** are satisfied, then

$$\forall \varepsilon > 0\ \forall t \geq 0\ \exists \delta_{\varepsilon,t} \in \left(0, \alpha_{M,t}\right) \forall (x,y) \in Z(t) \setminus \{0\}$$

$$\left(d(x, \mathbb{N}^t) \geq \varepsilon \Rightarrow \beta\left(x, y, \bar{p}(t)\right) \leq \alpha_{M,t} - \delta_{\varepsilon,t}\right)$$

or equivalently:

$$d(x, \mathbb{N}^t) \geq \varepsilon \Rightarrow \langle \bar{p}(t), y \rangle - \left(\alpha_{M,t} - \delta_{\varepsilon,t}\right) \langle \bar{p}(t), x \rangle \leq 0 \tag{9}$$

Proof is similar to the proof of Lemma 2 in Panek (2019a).[6] ∎

Under conditions **(G1)–(G6)**, without any additional restraints, it may happen that for a given $\varepsilon > 0$ (from (9)):

$$\delta_{\varepsilon,t} \longrightarrow 0 \quad \text{when} \quad t \longrightarrow +\infty$$

This condition means that the economic efficiency of a production process $(x, y)Z(t) \setminus \{0\}$ may increase with time, approaching the maximum level

$$\beta\left(x, y, \bar{p}(t)\right) \longrightarrow \alpha_{M,t} \quad \text{when} \quad t \longrightarrow +\infty$$

although the production/input structure in such processes will constantly deviate by ε from the (optimal) production structure that is achieved only in the turnpike. We can exclude this unrealistic situation by assuming, as in Panek (2019a), that:

(G7) $$\forall \varepsilon > 0\ \exists \nu_\varepsilon > 0\ \forall t \left(\frac{\delta_{\varepsilon,t}}{\alpha_{M,t}} \geq \nu_\varepsilon\right)$$

We assume that, regardless of the time horizon T, temporary equilibrium prices are non-increasing and limited:[7]

(G8) $$\exists \rho > 0\ \forall t_1 > 0\ \forall t < t_1 \left(\bar{p}(t) \geqq \bar{p}(t+1) \geq 0 \wedge \|\bar{p}(t)\| \leq \rho\right)$$

Temporary equilibrium prices $\bar{p}(t)$ are defined up to the structure. Thus, for this condition to occur, it is enough that they are positive. This assumption is realistic.

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[6] See also Theorem 5 in Panek (2016).

[7] See Panek (2014). Regarding the assumed monotonicity of the temporary equilibrium price trajectory, rather than assumption $\forall t < t_1 (\|\bar{p}(t)\| \leq \rho)$, it is sufficient to assume the weaker condition: $\|\bar{p}(0)\| \leq \rho$.

3. Feasible and optimal growth processes. 'Weak' turnpike effect

Let us determine a finite set of time periods $T = \{0,1, \dots, t_1\}$. Traditionally we call it the economy horizon. Usually we assume that the inputs $x(t+1)$ that are used in the economy in the next period, come from the outputs $y(t)$ produced in the previous period, $x(t+1) \leqq y(t)$, $t = 0,1, \dots, t_1 - 1$. That, together with condition **(G3)**, leads to the condition

$$\big(y(t), y(t+1)\big) \in Z(t+1), \quad t = 0,1, \dots, t_1 - 1 \tag{10}$$

An initial positive production vector y^0 in the period $t = 0$ is determined:

$$y(0) = y^0 > 0 \tag{11}$$

An economy that fulfils conditions **(G1)–(G8)**, (10), (11) is called a non--stationary Gale economy with a multilane turnpike and changing technology. Each production vector sequence $\{y(t)\}_{t=0}^{t_1}$ satisfying conditions (10)–(11) is called (y^0, t_1) – feasible growth process (production trajectory).

Let $u: R_+^n \longrightarrow R^1$ denote a continuous, concave and increasing utility function, positively homogeneous of degree 1, that is determined on production vectors in the final period t_1 of the time horizon T and satisfying the following conditions:

(G9)(i) $\qquad \exists a > 0 \ \ \forall s \in S_+^n(1)\big(u(s) \leq a\langle \bar{p}(t_1), s\rangle\big)$

where

$$S_+^n(1) = \{x \in \mathbb{R}_+^n \,|\, \|x\| = 1\}$$

(2i) $\qquad \forall s \in S(0)(u(s) > 0)$[8]

We denote the feasible growth process that is a solution to the following target growth problem (maximising the utility of the outputs produced in the last period of time horizon T):

$$\max u\big(y(t_1)\big)$$

[8] Condition **(i)** states that regardless of the length of the horizon T, the utility function may be approximated by the linear form with a coefficient vector $a\bar{p}(t_1)$; $a > 0$. Some of the CES utility functions that are positively homogeneous of degree 1 meet conditions **(i)** and **(2i)**.

$$\text{subject to (10), (11)} \tag{12}$$

$$\text{(with fixed } y^0\text{)}$$

by $\{y^*(t)\}_{t=0}^{t_1}$ and we call it the (y^0, t_1, u) – optimal growth process. Under our assumptions there exists a solution to the problem (12).[9]

The author's previous paper (Panek, 2019a) focused mainly on the Gale economy in which an output vector $\bar{y}(t)$ produced in each optimal process $(\bar{x}(t), \bar{y}(t)) \in Z_{opt}(t) \subset Z(t)$ in period t was at the same time an input vector $\bar{x}(t+1)$ in an optimal process $(\bar{x}(t+1), \bar{y}(t+1)) \in Z_{opt}(t+1) \subset Z(t+1)$ in period $t+1$. This required the assumption that the production technology available in the economy in period t would be available also in period $t+1$. This assumption, although natural in a short period of time, is difficult to justify in the longer term. Below, as in Panek (2019b), we formulate the following, much weaker condition:

(G10)
$$\exists\{\bar{x}(t), \bar{y}(t)\}_{t=0}^{t_1}\ \forall t \in T\left(\left(\bar{x}(t), \bar{y}(t)\right) \in Z_{opt}(t)\right) \wedge\ \forall t < t_1$$
$$\left(\bar{x}(t+1) = \bar{y}(t)\right)$$

According to **(G10)**, **(G3)**, (5) and (10), there exists at least one series of the optimal production processes $\{\bar{x}(t), \bar{y}(t)\}_{t=0}^{t_1}$, such that:

$$\frac{\bar{y}(0)}{\|\bar{y}(0)\|} = \bar{s} \in S(0), \left(\bar{y}(t), \bar{y}(t+1)\right) \in Z(t+1),\ \bar{y}(t+1) =$$
$$= \alpha_{M,t+1}\bar{y}(t),\ \ t = 0, \dots, t_1 - 1$$

That is,

$$\bar{y}(t) = \left(\prod_{\theta=1}^{t} \alpha_{M,\theta}\right)\bar{y}(0), \qquad t = 1, \dots, t_1 \tag{13}$$

and

$$\forall t \in T(\bar{y}(t) \in N_{\bar{s}}^0 = \{\lambda\bar{s} | \lambda > 0\} \in \mathbb{N}^0)$$

[9] This is equivalent to the problem $\max\limits_{y(t_1) \in R_{y^0,t_1}} u\left(y(t_1)\right)$ (maximizing continuous function $u(\cdot)$ on the compact set R_{y^0,t_1} of all the output vectors that it is possible to produce in period t_1 by an economy that 'started' in the initial period $t = 0$ from the state y^0). The existence of the solution follows from the Weierstrass theorem on the existence of the maximum of the continuous function on the compact set; the proof is similar to the proof of Th. 5.7 in Panek (2003; chapter 5).

The trajectory $\{\bar{y}(t)\}_{t=0}^{t_1}$ and any positive $\lambda -$ multiple of this trajectory lies entirely on the von Neumann ray $N_{\bar{s}}^0$: if $\bar{y}(t) \in N_{\bar{s}}^0$, thus for any $\lambda > 0$ we also get $\lambda\bar{y}(t) \in N_{\bar{s}}^0,\ t = 0, 1, \dots, t_1$. Since $\bar{s}(t) = \frac{\bar{y}(t)}{\|\bar{y}(t)\|} \in S(t)$ and $\frac{\bar{y}(t)}{\|\bar{y}(t)\|} = \frac{\bar{y}(0)}{\|\bar{y}(0)\|} = \bar{s} \in S(0)$, thus

$$\forall t \in T(N_{\bar{s}}^0 \in \mathbb{N}^t)$$

i.e. $N_{\bar{s}}^0$ (a single turnpike, possibly unique) belongs to the multilane turnpike $\mathbb{N}^t$ in all periods of the horizon T. We call it the peak von Neumann ray (peak turnpike) in a non-stationary Gale economy with changing technology. The economy achieves the highest growth rate $\alpha_{M,t}$ over the entire time horizon T (in all periods $t \in T$) only on the peak turnpike $N_{\bar{s}}^0$.[10]

In Panek (2019a, 2019b) we proved that in long periods (in a long horizon T) each $(y^0, t_1, u) -$ optimal process $\{y^*(t)\}_{t=0}^{t_1}$ almost always runs in close proximity to the widest of all the multilane turnpikes $\mathbb{N}^{t_1} = \bigcup_{t \in T} \mathbb{N}^t$ (the shape the turnpike adopts in the last period t_1 of the horizon T)[11] available in the economy. Theorem 3 formulated below describes the stability of the optimal growth processes in proximity to each of the multiline turnpikes $\mathbb{N}^t$ that exist in the economy in periods $t = 1, 2, \dots, t_1$ of the horizon T.

□ **Theorem 3.** If conditions **(G1)–(G10)** are met, then for any $\varepsilon > 0$ there exists a natural number k_ε, such that the number of time periods in which $(y^0, t_1, u) -$ optimal process $\{y^*(t)\}_{t=0}^{t_1}$ satisfies the condition[12]

$$d(y^*(t), \mathbb{N}^{t+1}) \geq \varepsilon \tag{14}$$

does not exceed k_ε. The number k_ε does not depend on the length of the horizon.

Proof. From the definition of the $(y^0, t_1, u) -$ optimal process, according to (6), (10) we obtain the following condition:

$$\langle \bar{p}(t+1), y^*(t+1) \rangle \leq \alpha_{M,t+1} \langle \bar{p}(t+1), y^*(t) \rangle,\ t = 0,1, \dots, t_1 - 1,$$

[10] We are only able to say about all the other rays (single turnpikes) N_s^t ($t \in T, s \in S(t), s \neq \bar{s}$), both those existing in the economy in period $t = 0$ and the one that will arise later, that among them there may (but does not have to) exist a single turnpike available in all the horizon T, as well as those appearing in a certain period $\tau_1 \geq 0$ and disappearing in a certain period $\tau_2 \leq t_1$.

[11] In the above-mentioned papers, the production spaces meet the condition $Z(t) \subseteq Z(t+1)$. Now we revoke this condition. This means that in the current version of the model, not every production technology available in the economy in period t also has to be available in the future; therefore inclusions $\mathbb{N}^t \subseteq \mathbb{N}^{t+1}$, $t < t_1$ might not happen. Regarding **(G10)**, we always get (regardless of the length of the horizon T): $\bigcap_{t=0}^{t_1} \mathbb{N}^t \neq \emptyset$ (because of our assumptions the entire ray $N_{\bar{s}}^0$ belongs to the set $\bigcap_{t=0}^{t_1} \mathbb{N}^t$).

[12] In previous papers (2019a, 2019b), instead of metrics $d(y^*(t), \mathbb{N}^{t+1})$, there is $d(y^*(t), \mathbb{N}^{t_1})$.

i.e. (considering **(G8)**):

$$\begin{gathered}\langle \bar{p}(t_1), y^*(t_1)\rangle \le \alpha_{M,t_1}\langle \bar{p}(t_1), y^*(t_1-1)\rangle \le \\ \le \alpha_{M,t_1}\langle \bar{p}(t_1-1), y^*(t_1-1)\rangle \le \\ \le \ \alpha_{M,t_1}\alpha_{M,t_1-1}\langle \bar{p}(t_1-1), y^*(t_1-2)\rangle \le \\ \dots \le \prod_{t=1}^{t_1}\alpha_{M,t}\langle \bar{p}(1), y^0\rangle \le \prod_{t=1}^{t_1}\alpha_{M,t}\langle \bar{p}(0), y^0\rangle\end{gathered} \tag{15}$$

Let us assume that in periods $\tau_1, \tau_2, \dots, \tau_k < t_1$, the inequality (14) holds. Let $L = \{\tau_1, \tau_2, \dots, \tau_k\}$. Then according to (9):

$$\langle \bar{p}(t+1), y^*(t+1)\rangle \le \left(\alpha_{M,t+1} - \delta_{\varepsilon,t+1}\right)\langle \bar{p}(t+1), y^*(t)\rangle,\ t \in L \tag{16}$$

From (15), (16) we get the condition:

$$\langle \bar{p}(t_1), y^*(t_1)\rangle \le \left(\prod_{\substack{t=1\\ t\notin L}}^{t_1}\alpha_{M,t+1}\right)\left(\prod_{t\in L}\left(\alpha_{M,t+1} - \delta_{\varepsilon,t+1}\right)\right)\langle \bar{p}(0), y^0\rangle$$

and thus, due to **(G9)(i)**, we reach the upper limit of the output utility produced in the last period of horizon T:

$$\begin{gathered}u\big(y^*(t_1)\big) \le a\langle \bar{p}(t_1), y^*(t_1)\rangle \le a\left(\prod_{\substack{t=1\\ t\notin L}}^{t_1}\alpha_{M,t+1}\right) \\ \left(\prod_{t\in L}\left(\alpha_{M,t+1} - \delta_{\varepsilon,t+1}\right)\right)\langle \bar{p}(0), y^0\rangle\end{gathered} \tag{17}$$

The initial production vector y^0 is positive (see (11)). If we take $\sigma = \min_i \frac{y_i^0}{\bar{s}_i} > 0$, then we are able to construct (y^0, t_1) – acceptable process $\{\tilde{y}(t)\}_{t=0}^{t_1}$:

$$\tilde{y}(t) = \begin{cases} y^0, & t = 0 \\ \sigma\left(\prod_{\theta=1}^{t}\alpha_{M,\theta}\right)\bar{s}, & t = 1, \dots, t_1 \end{cases}$$

in which an economy starting in the initial period from the state y^0 reaches the single (peak) turnpike $N_{\bar{s}}^0$, $\bar{s} \in S(0)$ already in the next period $t = 1$. The economy can remain on this turnpike until the end of the horizon T, see (13). Due to the positive homogeneity of the degree 1 of the utility function, from

(G9)(2i) and from the definition of the (y^0, t_1, u) – optimal process, the following inequality holds:

$$u(y^*(t_1)) \geq u(\tilde{y}(t_1)) = \sigma\left(\prod_{\theta=1}^{t_1} \alpha_{M,\theta}\right) u(\bar{s}) > 0 \tag{18}$$

By combining (17) and (18), we obtain the condition:

$$a\left(\prod_{\substack{t=1\\ t\notin L}}^{t_1} \alpha_{M,t+1}\right)\left(\prod_{t\in L}(\alpha_{M,t+1} - \delta_{\varepsilon,t+1})\right)\langle \bar{p}(0), y^0\rangle \geq$$
$$\geq \sigma\left(\prod_{\theta=1}^{t_1} \alpha_{M,\theta}\right) u(\bar{s}) > 0$$

and hence after the transformations, regarding **(G8)** (and using the fact that $u(\cdot)$ is positive and continuous on the compact set $S(0)$), we obtain:

$$\prod_{t=\tau_1}^{\tau_k}\left(1 - \frac{\delta_{\varepsilon,t+1}}{\alpha_{M,t+1}}\right) \geq \frac{\sigma u(\bar{s})}{a\langle \bar{p}(0), y^0\rangle} \geq \frac{\sigma u_{min}}{a\rho y_{max}^0} = C > 0 \tag{19}$$

where $u_{min} = \min_{s\in S(0)} u(s) > 0$, $y_{max}^0 = \max_i y_i^0 > 0$

According to **(G7)**, there exists $\nu_\varepsilon > 0$, such that for any moment t we have $\frac{\delta_{\varepsilon,t}}{\alpha_{M,t}} \geq \nu_\varepsilon$, which in combination with condition (19) leads to the inequality $(1 - \nu_\varepsilon)^k \geq C$ and allows us to estimate k:

$$k \leq \frac{\ln C}{\ln(1 - \nu_\varepsilon)} = A \tag{20}$$

This is enough to take the smallest natural number not less than $\max\{0, k\}$ as k_ε. ■

4. Final remarks

Remark 1. Replacing condition (11) with a weaker condition:

$$y(0) = y^0 \geq 0 \tag{11'}$$

and with the assumption:

(G11) There exists $(y^0, \check{t})$ – acceptable growth process $\{\check{y}(t)\}_{t=0}^{\check{t}}$, $\check{t} < t_1$, in which[13]

$$\check{y}(\check{t}) > 0$$

we obtain the following version of the 'weak' multilane turnpike theorem:

□ **Theorem 4.** If conditions **(G1)–(G11)** are met and $\{y^*(t)\}_{t=0}^{t_1}$ is a (y^0, t_1, u) – optimal growth process (the solution to the problem (12) after replacing condition (11) with (11')), then for any number $\varepsilon > 0$ there exists a natural number k_ε such that the number of time periods, in which (y^0, t_1, u) – optimal process $\{y^*(t)\}_{t=0}^{t_1}$ satisfies the condition:

$$d(y^*(t), \mathbb{N}^{t+1}) \geq \varepsilon$$

does not exceed k_ε. The number k_ε does not depend on the length of the horizon T.

Proof. If we assume that (y^0, t_1, u) – optimal growth process $\{y^*(t)\}_{t=0}^{t_1}$ is the solution to the problem

$$\max\ u(y(t_1))$$

subject to (10), (11')

and by repeating the proof of Theorem 3, we come to the condition (17). If the condition **(G11)** is met, then there exists (y^0, t_1) – acceptable growth process $\{\tilde{y}(t)\}_{t=0}^{t_1}$:

$$\tilde{y}(t) = \begin{cases} \check{y}(t), & t = 0, 1, \dots \check{t} \\ \sigma\left(\prod_{\theta=\check{t}+1}^{t} \alpha_{M,\theta}\right)\bar{s}, & t = \check{t}+1, \dots, t_1 \end{cases} \tag{21}$$

$$\begin{gathered} \sigma = \min_i \frac{\check{y}_i(\check{t})}{\bar{s}_i} > 0 \text{ and} \\ u(y^*(t_1)) \geq u(\tilde{y}(t_1)) = \sigma\left(\prod_{\theta=\check{t}+1}^{t_1} \alpha_{M,\theta}\right) u(\bar{s}) > 0 \end{gathered} \tag{22}$$

[13] There exists $(y^0, \check{t})$ – acceptable growth process, in which the economy in period $\check{t}$ (before the end of the horizon T) is able to produce a positive output vector.

The rest of the proof is similar to the proof of Theorem 3. From (17), (22) we obtain the condition:

$$a\left(\prod_{\substack{t=1\\t\notin L}}^{t_1}\alpha_{M,t+1}\right)\left(\prod_{t\in L}\left(\alpha_{M,t+1}-\delta_{\varepsilon,t+1}\right)\right)\langle\bar{p}(0),y^0\rangle\geq$$
$$\geq\sigma\left(\prod_{\theta=\check{t}+1}^{t_1}\alpha_{M,\theta}\right)u(\bar{s})>0$$

in other words:

$$\prod_{t=\tau_1}^{\tau_k}\left(1-\frac{\delta_{\varepsilon,t+1}}{\alpha_{M,t+1}}\right)\geq\frac{\sigma u(\bar{s})}{a\langle\bar{p}(0),y^0\rangle\prod_{\theta=1}^{\check{t}}\alpha_{M,\theta}}\geq\frac{\sigma u_{min}}{a\rho y_{max}^0\prod_{\theta=1}^{\check{t}}\alpha_{M,\theta}}=C>0$$

Consequently, we obtain again the estimate (20). ■

Remark 2. Theorem 4 remains valid if we replace condition **(G11)** with the assumption that there exists $(y^0,\check{t})$ – feasible growth process $\{\check{y}(t)\}_{t=0}^{\check{t}}$, $\check{t}<t_1$, leading from the intitial state y^0 to the peak turnpike $\check{y}(\check{t})\in N_{\bar{s}}^0$. The proof proceeds in the same way as the proof of Theorem 4.

Remark 3. Theorem 4 also remains valid if – condition **(G10)** is replaced with a (weaker) condition:

(G10')
$$\exists\bar{t}<t_1\ \exists\{\bar{x}(t),\bar{y}(t)\}_{t=\bar{t}}^{t_1}\ \forall t\in\{\bar{t},\bar{t}+1,\dots,\ t_1\}$$
$$\left(\left(\bar{x}(t),\bar{y}(t)\right)\in Z_{opt}(t)\right)\ \wedge$$
$$\wedge\ \forall t\in\{\bar{t},\bar{t}+1,\dots,\ t_1-1\}\ \left(\bar{x}(t+1)=\bar{y}(t)\right)$$

and condition **(G11)** is replaced with the condition:

(G11') There exists $(y^0,\check{t})$ – acceptable growth process $\{\check{y}(t)\}_{t=0}^{\check{t}}$, $\check{t}<t_1$ leading in a period $\check{t}\geq\bar{t}$ from the initial state (11') to the peak turnpike $N_{\bar{s}}^{\bar{t}}=\{\lambda\bar{s}|\lambda>0\}$, where $\bar{s}=\frac{\bar{y}(\bar{t})}{\|\bar{y}(\bar{t})\|}\in S(\bar{t})$.[14]

Remark 4. In the model of the non-stationary economy with a multilane turnpike and changing production technology presented in this paper, the production growth rate $\alpha_{M,t}$ changes (increases) on the peak turnpike, whereas the production structure $\bar{s}(t)=\frac{\bar{y}(t)}{\|\bar{y}(t)\|}$ remains unchanged. In real economies it is not only the production growth rate that changes over time, but its structure as well. Sometimes these

[14] Then the peak turnpike $N_{\bar{s}}^{\bar{t}}$ exists from period $t=\bar{t}$ (not necessarily from $t=0$).

processes are very dynamic and are the result of technological and/or organisational progress, innovation, depletion of natural resources, changes in consumer demand, etc.

There is no evidence that analogous changes in the production structure cannot occur on the turnpike. Therefore, it seems reasonable to include the changing production structure on the peak turnpike in our non-stationary Gale economy with a multilane turnpike as well, which leads to the next stage of the research.[15]

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[15] In the case of a single (so called 'twisted') turnpike, some preliminary results include Keeler (1972), Panek (2015a, 2015b).

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