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# Multilane turnpike in the non-stationary input-output economy with von Neumann temporary equilibrium

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**Abstract.** In the author's earlier papers concerning asymptotic characteristics of the optimal growth processes in non-stationary Gale economies with multilane production turnpikes, it is assumed that production technology used in time period $t$ may also be used in the next period. Such an assumption, relevant for short periods, is difficult to justify in the longer term.

The paper contains a proof of the so called 'weak' effect of the multilane turnpike in a non--stationary Gale economy with changing technology, where this assumption has been suspended.

**Keywords:** non-stationary Gale economy, von Neumann temporary equilibrium, multilane turnpike

**JEL:** C62, C67, O41, O49

## 1. Basic assumptions and definitions

We assume that in the economy there are $n$ used and/or produced goods, where $n$ is an integer and is finite. It is assumed that the time $t$ is discrete, $t = 0, 1, 2, \ldots$ We let $x(t) = \left(x_1(t), \ldots, x_n(t)\right) \geqq 0$ denote goods that are used in period $t$ (an input vector or expenditure vector) and $y(t) = \left(y_1(t), \ldots, y_n(t)\right) \geqq 0$ goods that are produced in this period from inputs $x(t)$ (an output vector or production vector).[1] We say that $\left(x(t), y(t)\right)$ represents / describes a feasible production process (due to disposable technology in the economy in period $t$). We let $Z(t)$ denote the set of all technologically feasible production processes in period $t$ and we call it Gale's production space (technological set) in period $t$. The notation $(x, y) \in Z(t)$ (or equivalently $\left(x(t), y(t)\right) \in Z(t)$) means that the economy in period $t$ can produce the output $y(t)$ using inputs $x(t)$. Production spaces $Z(t), t = 0,1, \ldots$ are nonempty subsets of $\mathbb{R}^{2n}_+$, satisfying the following conditions:

**(G1)**
$$\forall (x^1, y^1) \in Z(t)\ \forall (x^2, y^2) \in Z(t)\ \forall\, \lambda_1, \lambda_2 \geq 0 \left(\lambda_1 (x^1, y^1) + \lambda_2 (x^2, y^2) \in Z(t)\right)$$

(homogeneity and additivity of production processes),

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[1] If $a \in R^n$, then $a \geqq 0$ means that $\forall i (a_i \geq 0)$, in contrast to $a \geq 0$, which means ($a \geqq 0 \wedge a \neq 0$).

**(G2)** $$\forall (x,y) \in Z(t)\ (x = 0 \Rightarrow y = 0)$$

(no land-of-Cockaigne condition),

**(G3)** $$\forall (x,y) \in Z(t)\ \forall x' \geqq x\ \ \forall\, 0 \leqq y' \leqq y\ \big((x',y') \in Z(t)\big)$$

(costless waste condition),

**(G4)** Production spaces $Z(t)$ are closed subsets of $\mathbb{R}^{2n}_{+}$.

From **(G1), (G2), (G4)** it follows that each production space $Z(t)$ is a closed convex cone in $\mathbb{R}^{2n}_{+}$ with a vertex at 0 and the property that if $(x,y) \in Z(t)$ and $(x,y) \neq 0$, then $x \neq 0$. We are only interested in such non-trivial processes $(x,y) \in Z(t)\backslash\{0\}$. These conditions imply that:

$$\forall t\ \forall (x,y) \in Z(t)\backslash\{0\}\ \exists \alpha(x,y) = \max\{\alpha |\ \alpha x \leqq y\} = \min_i \frac{y_i}{x_i} \geq 0$$

A non-negative number $\alpha(x,y)$ is called the technological efficiency rate of the process $(x,y)$. The function $\alpha(\cdot)$ is positively homogenous of degree zero on $Z(t)\backslash\{0\}$ and

$$\exists \big(\bar{x}(t), \bar{y}(t)\big) \in\ Z(t)\backslash\{0\}\ \Big(\alpha\big(\bar{x}(t), \bar{y}(t)\big) = \max_{(x,y)\in Z(t)\backslash\{0\}} \alpha(x,y) = \alpha_{M,t}\Big) \geq 0$$

see e.g. Takayama (1985, Th. 6.A.1; after replacing $Z(t)$ with $Z$ and $\alpha_{M,t}$ with $\alpha_M$). The process $\big(\bar{x}(t), \bar{y}(t)\big)$ is called an optimal production process while $\alpha_{M,t}$ an optimal efficiency rate in the non-stationary Gale economy in period $t$. Due to the positive homogeneity of degree zero of the function $\alpha(\cdot)$, an optimal production process multiplied by any positive number is also an optimal production process

$$\forall \lambda > 0 \Big(\ \alpha\big(\bar{x}(t), \bar{y}(t)\big) =\ \ \alpha\big(\lambda\bar{x}(t), \lambda\bar{y}(t)\big)\Big)$$

Let us assume that:

**(G5)** $$\forall t\ \forall i \in \{1,2,\ldots,n\}\ \exists (x,y) \in Z(t)\big((y_i) > 0\big)$$

(the economy has a technology that at any period $t$ allows the production of each good). This assumption along with **(G1)** ensures that the optimal efficiency rate $\alpha_{M,t}$ is always (at any period $t$) positive. Let us denote:

$$Z_{opt}(t) = \{(\bar{x}, \bar{y}) \in Z(t) \setminus \{0\} | \alpha(\bar{x}, \bar{y}) = \alpha_{M,t} > 0\}$$

This set consists of all the optimal production processes in a non-stationary Gale economy in period $t$. All sets $Z_{opt}(t) \subset Z(t), t = 0,1, \ldots$ are cones that are contained in $\mathbb{R}^{2n}_{+}$ (without 0).[2] From **(G3)** it follows that if $(\bar{x}, \bar{y}) \in Z_{opt}(t)$, then also $(\bar{x}, \alpha_{M,t}\bar{x}) \in Z_{opt}(t)$ and $(\bar{y}, \alpha_{M,t}\bar{y}) \in Z_{opt}(t)$. The vector $s(t) = \frac{\bar{y}(t)}{\|\bar{y}(t)\|}$ represents the production structure in the optimal process $(\bar{x}(t), \bar{y}(t)) = (\bar{x}, \bar{y}) \in Z(t) \setminus \{0\}$[3] Assuming **(G1)–(G5),** the sets[4]

$$S(t) = \left\{s(t) | \exists (\bar{x}(t), \bar{y}(t)) \in Z_{opt}(t) \left(s(t) = \frac{\bar{y}(t)}{\|\bar{y}(t)\|}\right)\right\}, t = 0, 1, \ldots$$

of the production structure vectors in all optimal processes in each period $t$ are nonempty, compact and convex; Panek (2016, Th. 2). If $s = s(t) \in S(t)$, then the ray

$$N_s^t = \{\lambda s | \lambda > 0\} \subset \mathbb{R}^n_{+}$$

is called a single production turnpike (von Neumann's ray) in a non-stationary Gale economy in period $t$. The set

$$\mathbb{N}^t = \bigcup_{s \in S(t)} N_s^t = \{\lambda s | \lambda > 0, s \in S(t)\}$$

of all the singular turnpikes $N_s^t$ forms a multilane production turnpike that is accessible in a non-stationary Gale model in period $t$. Each multilane turnpike $\mathbb{N}^t, t = 0,1, \ldots$, is a cone in $\mathbb{R}^n_{+}$ not containing 0.

In Panek (2018, Lemma 1) we prove that if in a non-stationary Gale economy satisfying conditions **(G1)–(G5)** in a certain period $t$ inputs structure $\frac{x}{\|x\|}$ or outputs structure $\frac{y}{\|y\|}$ in a process $(x, y) \in Z(t) \setminus \{0\}$ is different than a turnpike's structure, then its technological efficiency is lower than optimal:

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[2] See Panek (2016, Th. 1; after replacing $Z(t), \alpha_{M,t}$ with $\alpha_M$).

[3] Here and on, if $a \in \mathbb{R}^n$, then $\|a\| = \sum_{i=1}^{n} |a_i|, \frac{a}{\|a\|} = \left(\frac{a_1}{\|a\|}, \frac{a_2}{\|a\|}, \ldots, \frac{a_n}{\|a\|}\right)$

[4] Equivalently $S(t) = \left\{s(t) | \exists (\bar{x}(t), \bar{y}(t)) \in Z_{opt}(t) \left(s(t) = \frac{\bar{x}(t)}{\|\bar{x}(t)\|}\right)\right\}$.

$$\forall t\ \forall (x,y) \in Z(t)\backslash\{0\}\left(\frac{x}{\|x\|} \notin S(t) \vee \frac{y}{\|y\|} \notin S(t) \Rightarrow \alpha(x,y) < \alpha_{M,t}\right) \quad (1)$$

This condition is important in the proofs of turnpike theorems (Theorems 3 and 4).

Let $p(t) = (p_1(t), \dots , p_n(t)) \geq 0$ denote the price vector in an economy in period $t$. If $(x(t), y(t)) \in Z(t)\backslash\{0\}$, then $\langle p(t), x(t)\rangle = \sum_{i=1}^{n} p_i(t)x_i(t)$ represents the value of expenditures incurred in that economy in period $t$ (expressed in prices $p(t)$), while $\langle p(t), y(t)\rangle = \sum_{i=1}^{n} p_i(t)y_i(t)$ is the value of outputs produced in period $t$ from inputs $x(t)$. Let

$$\beta(x(t), y(t), p(t)) = \frac{\langle p(t), y(t)\rangle}{\langle p(t), x(t)\rangle} \geq 0$$

($\langle p(t), x(t)\rangle \neq 0$) denote the economic efficiency rate of the process $(x(t), y(t))$ in period $t$ (with prices $p(t)$). Let us take any optimal process $(\bar{x}(t), \bar{y}(t)) \in Z_{opt}(t)$. Then

$$\alpha_{M,t}\bar{x}(t) \leqq \bar{y}(t) \quad (2)$$

□ **Theorem 1.** Assuming **(G1)–(G5)**:

$$\forall t \geq 0\ \exists \bar{p}(t) \geq 0\ \ \forall (x,y) \in Z(t)\big(\langle \bar{p}(t), y\rangle \leq \alpha_{M,t}\langle \bar{p}(t), x\rangle\big) \quad (3)$$

**Proof.**[5] According to **(G1)** for any $t$ we have $(0,0) \in Z(t)$ and hence, regarding **(G3)**,

$$\forall t\ \forall i \in \{1, 2, \dots, n\}\left((e^i, 0) \in Z(t)\right)$$

where $e^i = (0, \dots, 1, \dots, 0) \in \mathbb{R}^n$ is an $n$-dimmensional vector with 1 as the $i$-th coordinate.

Let us choose any period $t$ and let us define the set

$$C(t) = \{c \mid \forall (x,y) \in Z(t)\ (c = \alpha_{M,t}x - y)\}$$

[5] Proof based on Panek (2019b, Th. 1).

This set is a convex cone in $\mathbb{R}^n$ (as a linear image of a cone $Z(t)$), which does not contain negative vectors. Indeed, if $C(t)$ contained a negative vector $c'$, then there would exist a production process $(x', y') \in Z(t)$, such that $c' = \alpha_{M,t}x' - y' < 0$, and therefore $\alpha_{M,t}x' < y'$. This would mean that

$$\exists \varepsilon' > 0 \left( \alpha_{M,t} = \max_{(x,y)\in Z(t)\setminus\{0\}} \alpha(x,y) \geq \alpha(x',y') \geq \alpha_{M,t} + \varepsilon' \right)$$

and this contradicts the definition of the optimal efficiency rate $\alpha_{M,t}$.

Since $(e^i, 0), i = 1,2, \dots, n$ belongs to $Z(t)$, therefore vectors

$$c^i = \alpha_{M,t}e^i - 0 = (0, \dots, \alpha_{M,t}, \dots, 0), \qquad i = 1, 2, \dots, n$$

belong to $C(t)$ (with $\alpha_{M,t} > 0$ on the $i$th coordinate). Then, the hyperplane separation theorem implies:

$$\exists \bar{p}(t) \neq 0 \ \forall c \in C(t)(\langle \bar{p}(t), c\rangle \geq 0)$$

In particular,

$$\langle \bar{p}(t), c^i \rangle = \alpha_{M,t}\bar{p}_i(t) \geq 0, \qquad i = 1, 2, \dots, n$$

so $\bar{p}(t) \geq 0$. Hence, condition (3) follows from $c = \alpha_{M,t}x - y$. ■

## 2. Temporary von Neumann equilibrium

It follows from (2) and (3) that

$$\forall (\bar{x}(t), \bar{y}(t)) \in Z_{opt}(t)\big( \langle \bar{p}(t), \bar{y}(t)\rangle = \alpha_{M,t}\langle \bar{p}(t), \bar{x}(t)\rangle \geq 0\big)$$

which means that it is theoretically possible that $\langle \bar{p}(t), \bar{y}(t)\rangle = 0$ (zero production value in the optimal process $(\bar{x}(t), \bar{y}(t))$. This unrealistic case does not occur when the following condition is met in the economy:

**(G6)**
$$\forall t \geq 0 \ \forall (x,y) \in Z(t)\setminus \{0\}\left( \alpha(x,y) < \alpha_{M,t} \Rightarrow 0 \leq \beta(x,y,\bar{p}(t)) = \frac{\langle \bar{p}(t), y\rangle}{\langle \bar{p}(t), x\rangle} < \alpha_{M,t} \right)$$

(no production process that does not have the highest technological efficiency can achieve the maximum economic efficiency).

□ **Theorem 2.** If $\bar{p}(t)$ are prices from Theorem 1, and conditions **(G1)–(G6)** are satisfied, then

$$\forall t \, \forall\big(\bar{x}(t), \bar{y}(t)\big) \in Z_{opt}(t) \subset Z(t) \; (\bar{p}(t), \bar{y}(t) = \\ = \alpha_{M,t}\langle \bar{p}(t), \bar{x}(t)\rangle > 0) \tag{4}$$

**Proof** is similar to the proof of Theorem 1 in Panek (2018) (after substituting $\big(\bar{x}(t), \bar{y}(t)\big) \in Z_{opt}(t)$, $\alpha_{M,t}$ instead of $(\bar{x}, \bar{y}) \in Z_{opt}$, $\alpha_M$). ■

When the optimal efficiency rate $\alpha_{M,t}$, optimal production process $\big(\bar{x}(t), \bar{y}(t)\big) \in$ $\in Z_{opt}(t) \subset Z(t)$ and prices $\bar{p}(t) \geq 0$ meet the following conditions:

$$\alpha_{M,t}\bar{x}(t) \leqq \bar{y}(t) \tag{5}$$

$$\forall (x, y) \in Z(t)\big(\langle \bar{p}(t), y\rangle \leq \alpha_{M,t}\langle \bar{p}(t), x\rangle\big) \tag{6}$$

$$\langle \bar{p}(t), \bar{y}(t)\rangle > 0 \tag{7}$$

we then say that they define the state of the temporary von Neumann equilibrium in a non-stationary Gale economy in period $t$. The vector $\bar{p}(t)$ is called the equilibrium price vector in period $t$. It follows from conditions (5)–(7) that in the equilibrium:

$$\beta\big(\bar{x}(t), \bar{y}(t), \bar{p}(t)\big) = \frac{\langle \bar{p}(t), \bar{y}(t)\rangle}{\langle \bar{p}(t), \bar{x}(t)\rangle} = \max_{(x,y)\in Z(t)\setminus\{0\}} \beta\big(x, y, \bar{p}(t)\big) = \\ = \alpha\big(\bar{x}(t), \bar{y}(t)\big) = \alpha_{M,t} > 0$$

i.e. in the temporary equilibrium in any period $t$, the economic efficiency of production equates with the technological efficiency, and in each case this is the highest possible efficiency for the economy. In the non-stationary Gale economy, under conditions **(G1)–(G6)**, the temporary equilibrium states always exist (in all periods of time). They are formed in each period $t$ by a triple $\{\alpha_{M,t}, \big(\bar{x}(t), \bar{y}(t)\big), \; \bar{p}(t)\}$ with any optimal production process $\big(\bar{x}(t), \bar{y}(t)\big) \in Z_{opt}(t)$. Prices vector $\bar{p}(t)$ and production process $\big(\bar{x}(t), \bar{y}(t)\big)$ in the temporary equilibrium are defined up to the structure (i.e. any positive multiple of a temporary equilibrium price vector is also an equilibrium price vector and any positive multiple of an optimal process is also optimal process).

Let us consider any production process $(x, y) \in Z(t)\setminus\{0\}$ and let

$$d(x, \mathbb{N}^t) = \inf_{x' \in \mathbb{N}^t} \left\| \frac{x}{\|x\|} - \frac{x'}{\|x'\|} \right\| \tag{8}$$

be a measure of the distance of the vector $x$ from the multilane turnpike $\mathbb{N}^t$ in period $t$. If conditions **(G1)–(G6)** are satisfied, then

$$\forall \varepsilon > 0\ \forall t \geq 0\ \exists \delta_{\varepsilon,t} \in \left(0, \alpha_{M,t}\right)\ \forall (x,y) \in Z(t) \setminus \{0\}$$

$$\left(d(x, \mathbb{N}^t) \geq \varepsilon \Rightarrow \beta\left(x, y, \bar{p}(t)\right) \leq \alpha_{M,t} - \delta_{\varepsilon,t}\right)$$

or equivalently:

$$d(x, \mathbb{N}^t) \geq \varepsilon \Rightarrow \langle \bar{p}(t), y \rangle - \left(\alpha_{M,t} - \delta_{\varepsilon,t}\right) \langle \bar{p}(t), x \rangle \leq 0 \tag{9}$$

**Proof** is similar to the proof of Lemma 2 in Panek (2019a).[6] ■

Under conditions **(G1)–(G6)**, without any additional restraints, it may happen that for a given $\varepsilon > 0$ (from (9)):

$$\delta_{\varepsilon,t} \longrightarrow 0 \quad \text{when} \quad t \longrightarrow +\infty$$

This condition means that the economic efficiency of a production process $(x, y) Z(t) \setminus \{0\}$ may increase with time, approaching the maximum level

$$\beta\left(x, y, \bar{p}(t)\right) \longrightarrow \alpha_{M,t} \quad \text{when} \quad t \longrightarrow +\infty$$

although the production/input structure in such processes will constantly deviate by $\varepsilon$ from the (optimal) production structure that is achieved only in the turnpike. We can exclude this unrealistic situation by assuming, as in Panek (2019a), that:

**(G7)** $$\forall \varepsilon > 0\ \exists \nu_\varepsilon > 0\ \forall t \left( \frac{\delta_{\varepsilon,t}}{\alpha_{M,t}} \geq \nu_\varepsilon \right)$$

We assume that, regardless of the time horizon $T$, temporary equilibrium prices are non-increasing and limited:[7]

**(G8)** $$\exists \rho > 0\ \forall t_1 > 0\ \forall t < t_1 (\bar{p}(t) \geqq \bar{p}(t+1) \geq 0 \wedge \|\bar{p}(t)\| \leq \rho)$$

Temporary equilibrium prices $\bar{p}(t)$ are defined up to the structure. Thus, for this condition to occur, it is enough that they are positive. This assumption is realistic.

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[6] See also Theorem 5 in Panek (2016).

[7] See Panek (2014). Regarding the assumed monotonicity of the temporary equilibrium price trajectory, rather than assumption $\forall t < t_1 (\|\bar{p}(t)\| \leq \rho)$, it is sufficient to assume the weaker condition: $\|\bar{p}(0)\| \leq \rho$.

## 3. Feasible and optimal growth processes. 'Weak' turnpike effect

Let us determine a finite set of time periods $T = \{0,1, \dots, t_1\}$. Traditionally we call it the economy horizon. Usually we assume that the inputs $x(t+1)$ that are used in the economy in the next period, come from the outputs $y(t)$ produced in the previous period, $x(t+1) \leqq y(t)$, $t = 0, 1, \dots, t_1 - 1$. That, together with condition **(G3)**, leads to the condition

$$\big(y(t), y(t+1)\big) \in Z(t+1), \quad t = 0, 1, \dots, t_1 - 1 \tag{10}$$

An initial positive production vector $y^0$ in the period $t = 0$ is determined:

$$y(0) = y^0 > 0 \tag{11}$$

An economy that fulfils conditions **(G1)–(G8)**, (10), (11) is called a non--stationary Gale economy with a multilane turnpike and changing technology. Each production vector sequence $\{y(t)\}_{t=0}^{t_1}$ satisfying conditions (10)–(11) is called $(y^0, t_1)$ – feasible growth process (production trajectory).

Let $u: R^n_+ \longrightarrow R^1$ denote a continuous, concave and increasing utility function, positively homogeneous of degree 1, that is determined on production vectors in the final period $t_1$ of the time horizon $T$ and satisfying the following conditions:

**(G9)(i)** $\qquad \exists a > 0 \ \ \forall s \in S^n_+(1)(u(s) \leq a\langle \bar{p}(t_1), s\rangle)$

where

$$S^n_+(1) = \{x \in \mathbb{R}^n_+ | \|x\| = 1\}$$

**(2i)** $\qquad \forall s \in S(0)(u(s) > 0)$[8]

We denote the feasible growth process that is a solution to the following target growth problem (maximising the utility of the outputs produced in the last period of time horizon $T$):

$$\max u\big(y(t_1)\big)$$

[8] Condition **(i)** states that regardless of the length of the horizon $T$, the utility function may be approximated by the linear form with a coefficient vector $a\bar{p}(t_1)$; $a > 0$. Some of the CES utility functions that are positively homogeneous of degree 1 meet conditions **(i)** and **(2i)**.

$$\text{subject to (10), (11)} \tag{12}$$

$$\text{(with fixed } y^0\text{)}$$

by $\{y^*(t)\}_{t=0}^{t_1}$ and we call it the $(y^0, t_1, u)$ – optimal growth process. Under our assumptions there exists a solution to the problem (12).[9]

The author's previous paper (Panek, 2019a) focused mainly on the Gale economy in which an output vector $\bar{y}(t)$ produced in each optimal process $(\bar{x}(t), \bar{y}(t)) \in Z_{opt}(t) \subset Z(t)$ in period $t$ was at the same time an input vector $\bar{x}(t+1)$ in an optimal process $(\bar{x}(t+1), \bar{y}(t+1)) \in Z_{opt}(t+1) \subset Z(t+1)$ in period $t+1$. This required the assumption that the production technology available in the economy in period $t$ would be available also in period $t+1$. This assumption, although natural in a short period of time, is difficult to justify in the longer term. Below, as in Panek (2019b), we formulate the following, much weaker condition:

**(G10)** $$\exists\{\bar{x}(t), \bar{y}(t)\}_{t=0}^{t_1}\ \forall t \in T\left((\bar{x}(t), \bar{y}(t)) \in Z_{opt}(t)\right) \wedge\ \forall t < t_1\ (\bar{x}(t+1) = \bar{y}(t))$$

According to **(G10)**, **(G3)**, (5) and (10), there exists at least one series of the optimal production processes $\{\bar{x}(t), \bar{y}(t)\}_{t=0}^{t_1}$, such that:

$$\frac{\bar{y}(0)}{\|\bar{y}(0)\|} = \bar{s} \in S(0),\ (\bar{y}(t), \bar{y}(t+1)) \in Z(t+1),\ \bar{y}(t+1) = \\ = \alpha_{M,t+1}\bar{y}(t),\ \ t = 0, \ldots, t_1 - 1$$

That is,

$$\bar{y}(t) = \left(\prod_{\theta=1}^{t} \alpha_{M,\theta}\right)\bar{y}(0), \qquad t = 1, \ldots, t_1 \tag{13}$$

and

$$\forall t \in T(\bar{y}(t) \in N_{\bar{s}}^0 = \{\lambda\bar{s} | \lambda > 0\} \in \mathbb{N}^0)$$

[9] This is equivalent to the problem $\max\limits_{y(t_1) \in R_{y^0,t_1}} u(y(t_1))$ (maximizing continuous function $u(\cdot)$ on the compact set $R_{y^0,t_1}$ of all the output vectors that it is possible to produce in period $t_1$ by an economy that 'started' in the initial period $t = 0$ from the state $y^0$). The existence of the solution follows from the Weierstrass theorem on the existence of the maximum of the continuous function on the compact set; the proof is similar to the proof of Th. 5.7 in Panek (2003; chapter 5).

The trajectory $\{\bar{y}(t)\}_{t=0}^{t_1}$ and any positive $\lambda -$ multiple of this trajectory lies entirely on the von Neumann ray $N_{\bar{s}}^0$: if $\bar{y}(t) \in N_{\bar{s}}^0$, thus for any $\lambda > 0$ we also get $\lambda\bar{y}(t) \in N_{\bar{s}}^0$, $t = 0, 1, \dots, t_1$. Since $\bar{s}(t) = \frac{\bar{y}(t)}{\|\bar{y}(t)\|} \in S(t)$ and $\frac{\bar{y}(t)}{\|\bar{y}(t)\|} = \frac{\bar{y}(0)}{\|\bar{y}(0)\|} = \bar{s} \in S(0)$, thus

$$\forall t \in T(N_{\bar{s}}^0 \in \mathbb{N}^t)$$

i.e. $N_{\bar{s}}^0$ (a single turnpike, possibly unique) belongs to the multilane turnpike $\mathbb{N}^t$ in all periods of the horizon $T$. We call it the peak von Neumann ray (peak turnpike) in a non-stationary Gale economy with changing technology. The economy achieves the highest growth rate $\alpha_{M,t}$ over the entire time horizon $T$ (in all periods $t \in T$) only on the peak turnpike $N_{\bar{s}}^0$.[10]

In Panek (2019a, 2019b) we proved that in long periods (in a long horizon $T$) each $(y^0, t_1, u) -$ optimal process $\{y^*(t)\}_{t=0}^{t_1}$ almost always runs in close proximity to the widest of all the multilane turnpikes $\mathbb{N}^{t_1} = \bigcup_{t \in T} \mathbb{N}^t$ (the shape the turnpike adopts in the last period $t_1$ of the horizon $T$)[11] available in the economy. Theorem 3 formulated below describes the stability of the optimal growth processes in proximity to each of the multiline turnpikes $\mathbb{N}^t$ that exist in the economy in periods $t = 1, 2, \dots, t_1$ of the horizon $T$.

□ **Theorem 3.** If conditions **(G1)–(G10)** are met, then for any $\varepsilon > 0$ there exists a natural number $k_\varepsilon$, such that the number of time periods in which $(y^0, t_1, u) -$ optimal process $\{y^*(t)\}_{t=0}^{t_1}$ satisfies the condition[12]

$$d(y^*(t), \mathbb{N}^{t+1}) \geq \varepsilon \tag{14}$$

does not exceed $k_\varepsilon$. The number $k_\varepsilon$ does not depend on the length of the horizon.

**Proof.** From the definition of the $(y^0, t_1, u) -$ optimal process, according to (6), (10) we obtain the following condition:

$$\langle \bar{p}(t+1), y^*(t+1) \rangle \leq \alpha_{M,t+1} \langle \bar{p}(t+1), y^*(t) \rangle,\ t = 0,1, \dots, t_1 - 1,$$

[10] We are only able to say about all the other rays (single turnpikes) $N_s^t$ ($t \in T, s \in S(t), s \neq \bar{s}$), both those existing in the economy in period $t = 0$ and the one that will arise later, that among them there may (but does not have to) exist a single turnpike available in all the horizon $T$, as well as those appearing in a certain period $\tau_1 \geq 0$ and disappearing in a certain period $\tau_2 \leq t_1$.

[11] In the above-mentioned papers, the production spaces meet the condition $Z(t) \subseteq Z(t+1)$. Now we revoke this condition. This means that in the current version of the model, not every production technology available in the economy in period $t$ also has to be available in the future; therefore inclusions $\mathbb{N}^t \subseteq \mathbb{N}^{t+1}$, $t < t_1$ might not happen. Regarding **(G10)**, we always get (regardless of the length of the horizon $T$): $\bigcap_{t=0}^{t_1} \mathbb{N}^t \neq \emptyset$ (because of our assumptions the entire ray $N_{\bar{s}}^0$ belongs to the set $\bigcap_{t=0}^{t_1} \mathbb{N}^t$).

[12] In previous papers (2019a, 2019b), instead of metrics $d(y^*(t), \mathbb{N}^{t+1})$, there is $d(y^*(t), \mathbb{N}^{t_1})$.

i.e. (considering **(G8)**):

$$\begin{aligned}\langle \bar{p}(t_1), y^*(t_1)\rangle &\le \alpha_{M,t_1}\langle \bar{p}(t_1), y^*(t_1-1)\rangle \le \\ &\le \alpha_{M,t_1}\langle \bar{p}(t_1-1), y^*(t_1-1)\rangle \le \\ &\le \alpha_{M,t_1}\alpha_{M,t_1-1}\langle \bar{p}(t_1-1), y^*(t_1-2)\rangle \le \\ \ldots &\le \prod_{t=1}^{t_1} \alpha_{M,t}\langle \bar{p}(1), y^0\rangle \le \prod_{t=1}^{t_1} \alpha_{M,t}\langle \bar{p}(0), y^0\rangle\end{aligned} \tag{15}$$

Let us assume that in periods $\tau_1, \tau_2, \ldots, \tau_k < t_1$, the inequality (14) holds. Let $L = \{\tau_1, \tau_2, \ldots, \tau_k\}$. Then according to (9):

$$\langle \bar{p}(t+1), y^*(t+1)\rangle \le \left(\alpha_{M,t+1} - \delta_{\varepsilon,t+1}\right)\langle \bar{p}(t+1), y^*(t)\rangle,\ t \in L \tag{16}$$

From (15), (16) we get the condition:

$$\langle \bar{p}(t_1), y^*(t_1)\rangle \le \left(\prod_{\substack{t=1 \\ t\notin L}}^{t_1} \alpha_{M,t+1}\right)\left(\prod_{t\in L}\left(\alpha_{M,t+1} - \delta_{\varepsilon,t+1}\right)\right)\langle \bar{p}(0), y^0\rangle$$

and thus, due to **(G9)(i)**, we reach the upper limit of the output utility produced in the last period of horizon $T$:

$$\begin{aligned}u\left(y^*(t_1)\right) \le a\langle \bar{p}(t_1), y^*(t_1)\rangle \le a\left(\prod_{\substack{t=1 \\ t\notin L}}^{t_1} \alpha_{M,t+1}\right) \\ \left(\prod_{t\in L}\left(\alpha_{M,t+1} - \delta_{\varepsilon,t+1}\right)\right)\langle \bar{p}(0), y^0\rangle\end{aligned} \tag{17}$$

The initial production vector $y^0$ is positive (see (11)). If we take $\sigma = \min_i \frac{y_i^0}{\bar{s}_i} > 0$, then we are able to construct $(y^0, t_1)$ − acceptable process $\{\tilde{y}(t)\}_{t=0}^{t_1}$:

$$\tilde{y}(t) = \begin{cases} y^0, & t = 0 \\ \sigma\left(\prod_{\theta=1}^{t} \alpha_{M,\theta}\right)\bar{s}, & t = 1, \ldots, t_1 \end{cases}$$

in which an economy starting in the initial period from the state $y^0$ reaches the single (peak) turnpike $N_{\bar{s}}^0$, $\bar{s} \in S(0)$ already in the next period $t = 1$. The economy can remain on this turnpike until the end of the horizon $T$, see (13). Due to the positive homogeneity of the degree 1 of the utility function, from

**(G9)(2i)** and from the definition of the $(y^0, t_1, u)$ − optimal process, the following inequality holds:

$$u(y^*(t_1)) \geq u(\tilde{y}(t_1)) = \sigma\left(\prod_{\theta=1}^{t_1} \alpha_{M,\theta}\right) u(\bar{s}) > 0 \tag{18}$$

By combining (17) and (18), we obtain the condition:

$$a\left(\prod_{\substack{t=1\\ t\notin L}}^{t_1} \alpha_{M,t+1}\right)\left(\prod_{t\in L}(\alpha_{M,t+1} - \delta_{\varepsilon,t+1})\right)\langle \bar{p}(0), y^0\rangle \geq$$
$$\geq \sigma\left(\prod_{\theta=1}^{t_1} \alpha_{M,\theta}\right) u(\bar{s}) > 0$$

and hence after the transformations, regarding **(G8)** (and using the fact that $u(\cdot)$ is positive and continuous on the compact set $S(0)$), we obtain:

$$\prod_{t=\tau_1}^{\tau_k}\left(1 - \frac{\delta_{\varepsilon,t+1}}{\alpha_{M,t+1}}\right) \geq \frac{\sigma u(\bar{s})}{a\langle \bar{p}(0), y^0\rangle} \geq \frac{\sigma u_{min}}{a\rho y^0_{max}} = C > 0 \tag{19}$$

where $u_{min} = \min_{s\in S(0)} u(s) > 0$, $y^0_{max} = \max_i y^0_i > 0$

According to **(G7)**, there exists $\nu_\varepsilon > 0$, such that for any moment $t$ we have $\frac{\delta_{\varepsilon,t}}{\alpha_{M,t}} \geq \nu_\varepsilon$, which in combination with condition (19) leads to the inequality $(1-\nu_\varepsilon)^k \geq C$ and allows us to estimate $k$:

$$k \leq \frac{\ln C}{\ln(1-\nu_\varepsilon)} = A \tag{20}$$

This is enough to take the smallest natural number not less than $\max\{0, k\}$ as $k_\varepsilon$.
■

## 4. Final remarks

*Remark 1.* Replacing condition (11) with a weaker condition:

$$y(0) = y^0 \geq 0 \tag{11'}$$

and with the assumption:

**(G11)** There exists $(y^0, \check{t})$ – acceptable growth process $\{\check{y}(t)\}_{t=0}^{\check{t}}$, $\check{t} < t_1$, in which[13]

$$\check{y}(\check{t}) > 0$$

we obtain the following version of the 'weak' multilane turnpike theorem:

□ **Theorem 4.** If conditions **(G1)–(G11)** are met and $\{y^*(t)\}_{t=0}^{t_1}$ is a $(y^0, t_1, u)$ – optimal growth process (the solution to the problem (12) after replacing condition (11) with (11')), then for any number $\varepsilon > 0$ there exists a natural number $k_\varepsilon$ such that the number of time periods, in which $(y^0, t_1, u)$ – optimal process $\{y^*(t)\}_{t=0}^{t_1}$ satisfies the condition:

$$d(y^*(t), \mathbb{N}^{t+1}) \geq \varepsilon$$

does not exceed $k_\varepsilon$. The number $k_\varepsilon$ does not depend on the length of the horizon $T$.

**Proof.** If we assume that $(y^0, t_1, u)$ – optimal growth process $\{y^*(t)\}_{t=0}^{t_1}$ is the solution to the problem

$$\max\ u(y(t_1))$$

$$\text{subject to (10), (11')}$$

and by repeating the proof of Theorem 3, we come to the condition (17). If the condition **(G11)** is met, then there exists $(y^0, t_1)$ – acceptable growth process $\{\tilde{y}(t)\}_{t=0}^{t_1}$:

$$\tilde{y}(t) = \begin{cases} \check{y}(t), & t = 0, 1, \dots \check{t} \\ \sigma\left(\prod_{\theta=\check{t}+1}^{t} \alpha_{M,\theta}\right)\bar{s}, & t = \check{t}+1, \dots, t_1 \end{cases} \tag{21}$$

$$\sigma = \min_i \frac{\check{y}_i(\check{t})}{\bar{s}_i} > 0 \text{ and}$$
$$u(y^*(t_1)) \geq u(\tilde{y}(t_1)) = \sigma\left(\prod_{\theta=\check{t}+1}^{t_1} \alpha_{M,\theta}\right) u(\bar{s}) > 0 \tag{22}$$

[13] There exists $(y^0, \check{t})$ – acceptable growth process, in which the economy in period $\check{t}$ (before the end of the horizon $T$) is able to produce a positive output vector.

The rest of the proof is similar to the proof of Theorem 3. From (17), (22) we obtain the condition:

$$a\left(\prod_{\substack{t=1\\t\notin L}}^{t_1}\alpha_{M,t+1}\right)\left(\prod_{t\in L}\left(\alpha_{M,t+1}-\delta_{\varepsilon,t+1}\right)\right)\langle\bar{p}(0),y^0\rangle\geq$$
$$\geq\sigma\left(\prod_{\theta=\check{t}+1}^{t_1}\alpha_{M,\theta}\right)u(\bar{s})>0$$

in other words:

$$\prod_{t=\tau_1}^{\tau_k}\left(1-\frac{\delta_{\varepsilon,t+1}}{\alpha_{M,t+1}}\right)\geq\frac{\sigma u(\bar{s})}{a\langle\bar{p}(0),y^0\rangle\prod_{\theta=1}^{\check{t}}\alpha_{M,\theta}}\geq\frac{\sigma u_{min}}{a\rho y_{max}^0\prod_{\theta=1}^{\check{t}}\alpha_{M,\theta}}=C>0$$

Consequently, we obtain again the estimate (20). ■

*Remark 2.* Theorem 4 remains valid if we replace condition **(G11)** with the assumption that there exists $(y^0,\check{t})$ – feasible growth process $\{\check{y}(t)\}_{t=0}^{\check{t}}$, $\check{t}<t_1$, leading from the intitial state $y^0$ to the peak turnpike $\check{y}(\check{t})\in N_{\bar{s}}^0$. The proof proceeds in the same way as the proof of Theorem 4.

*Remark 3.* Theorem 4 also remains valid if – condition **(G10)** is replaced with a (weaker) condition:

**(G10')**
$$\exists\bar{t}<t_1\ \exists\{\bar{x}(t),\bar{y}(t)\}_{t=\bar{t}}^{t_1}\ \forall t\in\{\bar{t},\bar{t}+1,\ldots,\ t_1\}$$
$$\left(\left(\bar{x}(t),\bar{y}(t)\right)\in Z_{opt}(t)\right)\ \wedge$$
$$\wedge\ \forall t\in\{\bar{t},\bar{t}+1,\ldots,\ t_1-1\}\ \left(\bar{x}(t+1)=\bar{y}(t)\right)$$

and condition **(G11)** is replaced with the condition:

**(G11')** There exists $(y^0,\check{t})$ – acceptable growth process $\{\check{y}(t)\}_{t=0}^{\check{t}}$, $\check{t}<t_1$ leading in a period $\check{t}\geq\bar{t}$ from the initial state (11') to the peak turnpike $N_{\bar{s}}^{\bar{t}}=\{\lambda\bar{s}|\lambda>0\}$, where $\bar{s}=\frac{\bar{y}(\bar{t})}{\|\bar{y}(\bar{t})\|}\in S(\bar{t})$.[14]

*Remark 4.* In the model of the non-stationary economy with a multilane turnpike and changing production technology presented in this paper, the production growth rate $\alpha_{M,t}$ changes (increases) on the peak turnpike, whereas the production structure $\bar{s}(t)=\frac{\bar{y}(t)}{\|\bar{y}(t)\|}$ remains unchanged. In real economies it is not only the production growth rate that changes over time, but its structure as well. Sometimes these

[14] Then the peak turnpike $N_{\bar{s}}^{\bar{t}}$ exists from period $t=\bar{t}$ (not necessarily from $t=0$).

processes are very dynamic and are the result of technological and/or organisational progress, innovation, depletion of natural resources, changes in consumer demand, etc.

There is no evidence that analogous changes in the production structure cannot occur on the turnpike. Therefore, it seems reasonable to include the changing production structure on the peak turnpike in our non-stationary Gale economy with a multilane turnpike as well, which leads to the next stage of the research.[15]

## References

Gale D., (1967), On optimal development in a multi-sector economy, *The Review of Economic Studies*, 34(1), 1–18. DOI: 10.2307/2296567.

Gantz D., (1980), A Strong Turnpike Theorem for a Nonstationary von Neumann-Gale Production Model, *Econometrica*, 48(7), 1777–1790. DOI: 10.2307/1911935.

Joshi S., (1997), Turnpike Theorems in Nonconvex Nonstationary Environments, *International Economic Review*, 38(1), 225–248. DOI: 10.2307/2527416.

Keeler E. B., (1972), A Twisted Turnpike, *International Economic Review*, 13(1), 160–166. DOI: 10.2307/2525912.

McKenzie L. W., (1976), Turnpike Theory, *Econometrica*, 44(5), 841–866. DOI: 10.2307/1911532.

McKenzie L. W., (1998), Turnpikes, *The American Economic Review*, 88(2), 1–14.

McKenzie L. W., (2005), Optimal Economic Growth, Turnpike Theorems and Comparative Dynamics, in: K. J. Arrow, M. D. Intriligator, (ed.), *Handbook of Mathematical Economics*, vol. 3, ed. 2, 1281–1355.

Makarov V. L., Rubinov A. M., (1977), *Mathematical Theory of Economic Dynmics and Equilibria*, Springer-Verlag, New York, Heidelberg, Berlin.

Nikaido H., (1968), *Convex Structures and Economic Theory*, Academic Press, New York.

Panek E., (2003), *Ekonomia matematyczna*, Publisher AE, Poznań.

Panek E., (2011), O pewnej wersji „słabego” twierdzenia o magistrali w modelu von Neumanna, *Przegląd Statystyczny*, 58(1–2), 75–87.

Panek E., (2014), Niestacjonarna gospodarka Gale’a z rosnącą efektywnością produkcji na magistrali, *Przegląd Statystyczny*, 61(1), 5–15.

Panek E., (2015a), Zakrzywiona magistrala w niestacjonarnej gospodarce Gale’a. Część I, *Przegląd Statystyczny*, 62(2), s. 149–163.

Panek E., (2015b), Zakrzywiona magistrala w niestacjonarnej gospodarce Gale’a. Część II, *Przegląd Statystyczny*, 62(4), s. 349–360.

Panek E., (2016), Gospodarka Gale’a z wieloma magistralami. „Słaby” efekt magistrali, *Przegląd Statystyczny*, 63(4), 355–374. DOI: 10.5604/01.3001.0014.1213.

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[15] In the case of a single (so called ‘twisted’) turnpike, some preliminary results include Keeler (1972), Panek (2015a, 2015b).

Panek E., (2017), „Słaby" efekt magistrali w niestacjonarnej gospodarce Gale'a z graniczną technologią i wielopasmową magistralą produkcyjną, in: D. Appenzeller, (ed.), *Matematyka i informatyka na usługach ekonomii*, Publisher UEP, Poznań, 94–110.

Panek E., (2018), Niestacjonarna gospodarka Gale'a z graniczną technologią i wielopasmową magistralą produkcyjną. „Słaby", „silny" i „bardzo silny" efekt magistrali, *Przegląd Statystyczny*, 65(4), 373–393. DOI: 10.5604/01.3001.0014.0595.

Panek E., (2019a), Optimal growth processes in non-stationary Gale economy with multilane production turnpike, *Economic and Business Review*, 5(2), 3–23. DOI: 10.18559/ebr.2019.2.1.

Panek E., (2019b), O pewnej wersji twierdzenia o wielopasmowej magistrali w niestacjonarnej gospodarce Gale'a, *Przegląd Statystyczny*, 66(2), 142–156. DOI: 10.5604/01.3001.0013.7608.

Takayama A., (1985), *Mathematical Economics*, Cambrige University Press, Cambridge.

Zalai E., (2004), The von Neumann Model and the Early Models of General Equilibrium, *Acta Oeconomica*, 54(1), 3–38. DOI: 10.1556/AOecon.54.2004.1.2.