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Substitution between production factors and intermediate inputs in the light of KLEMS growth accounting for Poland

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Abstract. The generally adopted view is that the gross-output-based MFP is the most correct in terms of methodology, and the value-added-based MFP is its imperfect substitute performed when some data are missing. In this paper, however, performing both of them and comparing their results is proposed as a valuable means to studying the development of outsourcing in the economy. The paper presents the elaboration of the methodology for the latter, which is its main contribution to the field. The case of the Polish economy is used as an applicative example (covering the period between 2005 and 2016), as KLEMS growth accounting has recently been implemented in Poland. The results demonstrate that around the year 2011, the expansion of outsourcing ceased. Since outsourcing was one of the main processes of the Polish transition, this observation can be considered as an indication of the maturing of the market economy in Poland. Moreover, KLEMS growth accounting makes it possible to study this issue through NACE activities, i.e. at the industry level. It shows that manufacturing (section C of NACE) is predominantly responsible for the situation described above, which is the main empirical finding of the study. The dominant role of manufacturing is also confirmed by some other sectoral observations of lesser importance. The methodology developed in this paper can potentially be applied to other countries for which both kinds of MFP are performed.

Keywords: gross value added, gross output, decomposition, production factors, KLEMS, productivity

JEL: O40, O47

1. Introduction

The article aims to discuss one particular aspect arising from the implementation of KLEMS growth accounting in Poland, and from the possibility to calculate both the value-added-based and the gross-output-based multifactor productivity (MFP). This aspect is a methodology developed in the paper for the purpose of comparing the two kinds of MFP and enabling the following discussion.

Although Poland is present in various releases of the EU KLEMS database, no decomposition of gross-value-added growth or gross-output growth into intermediate inputs contribution, primary production factor (i.e. labour and capital) contributions, or MFP contribution has ever been performed, because of insufficient input data (apart from 2007 EU KLEMS release, presently outdated). The reason is, on the one hand, that not enough data have been sent to Eurostat (although Poland

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has more data which theoretically could be sent, but they are not due to a partly voluntary character of co-operation agreements within Eurostat), and on the other, that innovative data imputation is sometimes necessary, as some data are not straightforwardly available in Poland. A growth accounting for Poland with a decomposition as mentioned above was performed by researchers appointed by the National Bank of Poland (NBP),[1] on the basis of a slightly different methodology (Gradzewicz et al., 2014, 2018), but not at the sectoral or industry level. To the authors' best knowledge, no one else has ever performed a decom-position of the above-mentioned kind at the industry level for Poland (apart from the KLEMS 2007 release).[2]

Lastly, some new source of input data, concerning intermediate inputs in prices, became available. This in turn allowed the performance of a gross output decomposition into the contributions of intermediate inputs, labour services, capital services and MFP, the latter being the gross-output-based type (as opposed to a gross-value-added-based type), also calculated residually. With that in mind, it is now possible to compare the two kinds of MFP, but on condition that the two computing regimes are consistent with each other. To meet this requirement, the paper presents as first the methodology adopted for the two types of the MFP calculation and their comparison.

The basic assumption of the study presented in the paper is that the two kinds of MFP should be exactly equal in the situation where there is no substitution between production factors and intermediate consumption, or, speaking precisely, one kind of MFP should be exactly convertible into the other through a standardised procedure shown e.g. in Timmer et al. (2007a, p. 16). However, in the real economy, this substitution happens. One of its forms is the possibility to outsource some activities instead of employing new persons, or even to replace existing employees with new outsourced services from external firms, so instead of the labour factor contribution growth we observe intermediate inputs growth. In such a situation, a difference appears between the two kinds of MFP; in other words, one kind of MFP is then not exactly convertible into the other kind. Since some services are provided by external firms, not only the labour factor is outsourced, but also the capital factor associated with given labour tasks. The capital factor can also be directly outsourced

[1] The Polish central bank.
[2] The EU KLEMS dataset release of 2007 includes a decomposition for Poland with labour services' contribution subdivided into hours worked and labour composition contributions, but with no subdivision of capital services' contribution into ICT and non-ICT capital contributions. The 2007 release covers the period of 1996–2004, so the time span directly preceding the time span of the present study. To be able to perform the former, data were often extensively imputed (Timmer et al., 2007b, pp. 121–129), to a far greater degree than in the present study (due to greater data shortages). The comparison of these two studies can possibly serve as the subject for further analysis.

by leasing. For example, instead of buying machines, they can be leased from external firms. Therefore, instead of capital contribution growth we observe intermediate inputs growth. This substitution effect is however observable to a lesser extent in statistics if there is vertical integration between companies in a given economy. Therefore, even though a signal observation can be provided that is not necessarily quantitatively comparable between different countries, it nevertheless is valuable. The initial hypothesis was that this signal observation is feasible thanks to the computation methodology proposed further.

When comparing the two kinds of MFP, it was assumed that they are both valuable analytical tools.[3] Suppose the gross output and intermediate consumption data are of good quality, and the tool effects associated with additional computations are negligible. In such a case, the additional procedure of gross output growth decomposition can generate significant analytical benefits related to the monitoring of outsourcing activities and to the monitoring of the blurred boundary between capital investments and intermediate consumption outlays (i.e. in the context of frequently changing accountancy and tax regulations and their random interpretations by the revenue administration, and due to some other related circumstances). Since the monitoring of outsourcing has a much stronger impact on the results, the analysis of the substitution between the contributions of primary production factors and the contribution of intermediate inputs involves mainly the analysis of the change in the scale of outsourcing deployment.[4]

The change in outsourcing is, however, particularly intensive when structural changes in an economy accelerate. For a transition economy or any economy undergoing major changes, outsourcing change should become more conspicuous. Therefore, appropriately devised comparisons between the gross-output-based MFP and the gross-value-added-based MFP can be used, to some extent, to trace a transition a given economy. The non-tool difference between the two MFPs can be considered a litmus test for the structural and market-oriented change. If no specific issues are involved, this assumption seems plausible, although strong. In the case of a transition economy like Poland, it seems particularly sensible to assume that the ceasing of the main structural (and other market-oriented) changes can be associated with the maturation of the market economy in this country, and this phenomenon can be traced, at least to some degree, by using the method of

[3] This is consistent with the Organisation for Economic Co-operation and Development (OECD, 2001, p. 31) and e.g. Phelps (2010). The problem is discussed more extensively in Schreyer and Pilat (2001, p. 129 and following) and e.g. Hall (1989).

[4] This is consistent with the OECD (2001, p. 29). Non-proportional technological change concerning the factors and intermediate consumption should also be taken into consideration here (OECD, 2001, p. 28), although to a lesser degree.

comparison between the two kinds of MFP. Moreover, this analysis can become interesting at the industry level.

The methodological framework for the comparison between the gross-output-based and the gross-value-added-based multifactor productivity is outlined in the second section of this paper. In the third section, these results are discussed in the context of the aggregate economy, and some interpretations are provided. In the fourth section a sample analysis at the industry level is presented. The fifth section consists of the conclusions. As they are debatable to a large extent, these outcomes remain open to further analyses and discussion.

2. The adopted methodology

The basic methodology for this study roots in the growth accounting methodology developed by Dale W. Jorgenson and associates, as outlined in Jorgenson (1963), Jorgenson and Griliches (1967), Jorgenson et al. (1987), Jorgenson (1989) and Jorgenson et al. (2005).[5] This underlying methodology has been summarised by Timmer et al. (2007a), and O'Mahony and Timmer (2009) for the EU KLEMS.[6] For Poland, it has been developed and presented in Kotlewski and Błażej (2018, 2020). From now on, only the basic formulae that will be referred to later will be provided. One of them concerns the standard decomposition of gross output growth into the contributions of intermediate inputs, production factor (labour and capital) services, and MFP:

$$\Delta\, ln\, Y_{jt} = \bar{v}^X_{jt} \Delta\, ln\, X_{jt} + \bar{v}^K_{jt}\, \Delta\, ln\, K_{jt} + \bar{v}^L_{jt} \Delta\, ln\, L_{jt} + \Delta\, ln\, A^Y_{jt}, \tag{1}$$

where Y is gross output, X – intermediate consumption, K – capital services,[7] L – labour services,[8] and A^Y stands for multifactor productivity (denominated as gross-output-based). These values are subscripted by j for industries and t for years.

[5] In the preparatory works for KLEMS implementation in Poland, the OECD growth accounting methodology was studied as well for possible insights; see OECD (2001, 2009, 2013) and Wölfl and Hajkova (2007).

[6] See also the overview of the subject: Jorgenson (2009).

[7] It is assumed that the values of capital services are proportional to the values of capital stocks if these are separated into different kinds of capital stocks at the industry level, which means that although capital stocks and capital services are different entities, their growths are assumed to be equal at this level. These different kinds of capital stocks are then aggregated by means of the Törnqvist quantity index at the industry level. Based on: OECD (2001, p. 61), Timmer et al. (2007a, p. 32–33), OECD (2009, p. 60) and Timmer et al. (2010, eq. (3.6)).

[8] It is assumed that the values of labour services are proportional to the amounts of physical work engaged (in hours worked), if it is divided into different kinds of labour according to age, level of education and sex. In the KLEMS framework there are 3 age levels, 3 education attainment levels and 2 sexes, which gives (3 x 3 x 2) 18 kinds of labour.

$\bar{v}$ with appropriate subscripts are average value shares[9] of the intermediate consumption and production factors in the gross output (defined in the superscripts by X, K and L) for two discrete periods $t-1$ and t, which are calculated through linear interpolation as $\bar{v} = (v_{t-1} + v_t)/2$ (for simplicity the subscripts of formula (1) have been omitted here). Since the growth of A^Y is residually calculated, equation (1) is consistently satisfied. In performing KLEMS growth accounting, the methodology is often reduced to a gross-value-added growth decomposition following the standard equation:

$$\Delta \, ln \, V_{jt} = \bar{w}_{jt}^{K} \Delta \, ln \, K_{jt} + \bar{w}_{jt}^{L} \Delta \, ln \, L_{jt} + \Delta \, ln \, A_{jt}^{V}, \tag{2}$$

where V is the gross value added and A^V stands for MFP (denominated as gross-value-added-based[10]). $\bar{w}$ with appropriate subscripts are average value shares of production factor services in gross value added (defined in the superscripts as K and L) for two discrete periods $t-1$ and t, which are calculated through linear interpolation in a similar way as $\bar{v}$ for the previous formula (1). The other symbols are the same as in Equation (1). Replacing the decomposition (1) by (2) solves some data problems and increases the international comparability between countries.[11] In practice, the contribution of MFP $\Delta \, ln \, A_{jt}^{V}$ is residually calculated as the subtraction between the other values, so Equation (2) is always satisfied, just like Equation (1). Therefore, there is no need to directly measure the levels of A in both of them.

The more universally performed (in the KLEMS growth accounting) decomposition of gross-value-added growth, as mentioned above in formula (2), can be extended into a decomposition of gross-output growth, as mentioned above in formula (1), on condition that the 'deflators' for the intermediate consumption are available also at the industry level – they are usually calculated as ratios between values expressed in current prices and values expressed in constant prices. For many countries (possibly for most of them), the decomposition of gross output growth based on formula (1) is not performed, while the growth decomposition based on formula (2) is, which results from the unavailability of some necessary data expressed in current and constant prices. For a few years, however, in Poland, the Department of National Accounts of Statistics Poland has published statistical data containing the information that allows the performance of the necessary calculations.

[9] All value shares referred to in the paper were taken from the national accounts, but they were adjusted for the self-employed before having been used in the calculations.
[10] It can be considered as a variant of total factor productivity (TFP).
[11] Because of different degrees of vertical integration of firms in different countries, which hinders the international comparability among the countries, as far as the intermediate consumption is considered.

To perform the calculations properly, they should remain consistent with the calculations already carried out for the gross-value-added growth decomposition, i.e., the values already calculated for this decomposition should be inserted into new formulae. Some mathematical tool discrepancies will then be reduced. To do so, some values from formula (2) have to be transposed into formula (1), as follows:

$$\Delta\, ln\, Y_{jt} = \bar{v}_{jt}^{X} \Delta\, ln\, X_{jt} + \overline{\left(\frac{V_{jt}}{Y_{jt}}\right)}\, \bar{w}_{jt}^{K}\, \Delta\, ln\, K_{jt} + \overline{\left(\frac{V_{jt}}{Y_{jt}}\right)}\, \bar{w}_{jt}^{L} \Delta\, ln\, L_{jt} + \Delta\, ln\, A_{jt}^{Y}. \qquad (3)$$

As can be seen in formula (3), the components taken from formula (2) are the components related to the primary production factor services, i.e. labour and capital services. These components must be multiplied by the ratios between gross value added and gross output at the j industry level. Moreover, they should be the averages for two discrete periods, $t-1$ and t, which are calculated through linear interpolation in a similar way to the shares for the previous formulae (1) and (2). The justification for the adoption of this linear interpolation is the same as for the shares, i.e., to make the approximation more precise.

The contributions of production factors services from formula (3) should therefore be further decomposed in KLEMS growth accounting as follows:

$$\overline{\left(\frac{V_{jt}}{Y_{jt}}\right)}\, \bar{w}_{jt}^{K} \Delta\, ln\, K_{jt} = \overline{\left(\frac{V_{jt}}{Y_{jt}}\right)}\, \bar{w}_{jt}^{KIT} \Delta\, ln\, KIT_{jt} + \overline{\left(\frac{V_{jt}}{Y_{jt}}\right)}\, \bar{w}_{jt}^{KNIT} \Delta\, ln\, KNIT_{jt}, \qquad (4)$$

$$\overline{\left(\frac{V_{jt}}{Y_{jt}}\right)}\, \bar{w}_{jt}^{L} \Delta\, ln\, L_{jt} = \overline{\left(\frac{V_{jt}}{Y_{jt}}\right)}\, \bar{w}_{jt}^{L} \Delta\, ln\, H_{jt} + \overline{\left(\frac{V_{jt}}{Y_{jt}}\right)}\, \bar{w}_{jt}^{L} \Delta\, ln\, LC_{jt}. \qquad (5)$$

In formula (4), *KIT* denotes the ICT capital and *KNIT* the non-ICT capital, whereas in formula (5), *H* represents the hours worked and *LC* the labour quality, otherwise called labour composition.

The contribution of MFP to the gross output relative growth (i.e. the contribution of gross-output-based MFP) from formula (1) and (3) can be made comparable with the contribution of MFP to the relative gross value added growth (i.e. the contribution of gross-value-added-based MFP) from formula (2), if it is multiplied by the inverse ratio between gross value added and gross output at the industry j level taken from formula (3):[12]

$$\Delta\, ln\, A_{jt}^{V*} = \overline{\left(\frac{Y_{jt}}{V_{jt}}\right)}\, \Delta\, ln\, A_{jt}^{Y}. \qquad (6)$$

[12] This is consistent with the OECD (2001, pp. 25–27) and Timmer et al. (2007a, p. 16).

The asterisk indicates that the value (A_{jt}^{V*}) from the left-hand side of formula (6) is the value derived from the gross-output-based MFP (A_{jt}^{Y}), which can be equal to gross-value-added-based MFP (A_{jt}^{V}) on condition that there is no substitution between the production factor services and intermediate consumption.[13] Then, if some mathematical tool discrepancies are ignored, the following approximation becomes abiding:

$$\Delta \, ln \, A_{jt}^{V*} \approx \Delta \, ln \, A_{jt}^{V}. \tag{7}$$

It means that the resulting value for the MFP contribution to the gross-value-added growth received from the conversion of gross-output-based MFP, from the left-hand side of formulae (6) and (7), should in principle be identical to the MFP residually calculated from the gross-value-added growth decomposition, from the right-hand side of formula (7). If it is not so, the phenomenon of the substitution between the primary[14] production factors and the intermediate consumption should be considered as substantial.

3. Discussion on the results

Suppose the above-mentioned substitution between the production factors (and more precisely, production factor services) and intermediate consumption is substantial. In such a case, it can be asserted that substantial changes are underway in the economy, as far as the outsourcing is considered. This concerns primarily the labour factor, but also, although to a lesser degree, the independent capital factor substitution by intermediate consumption should be considered here.

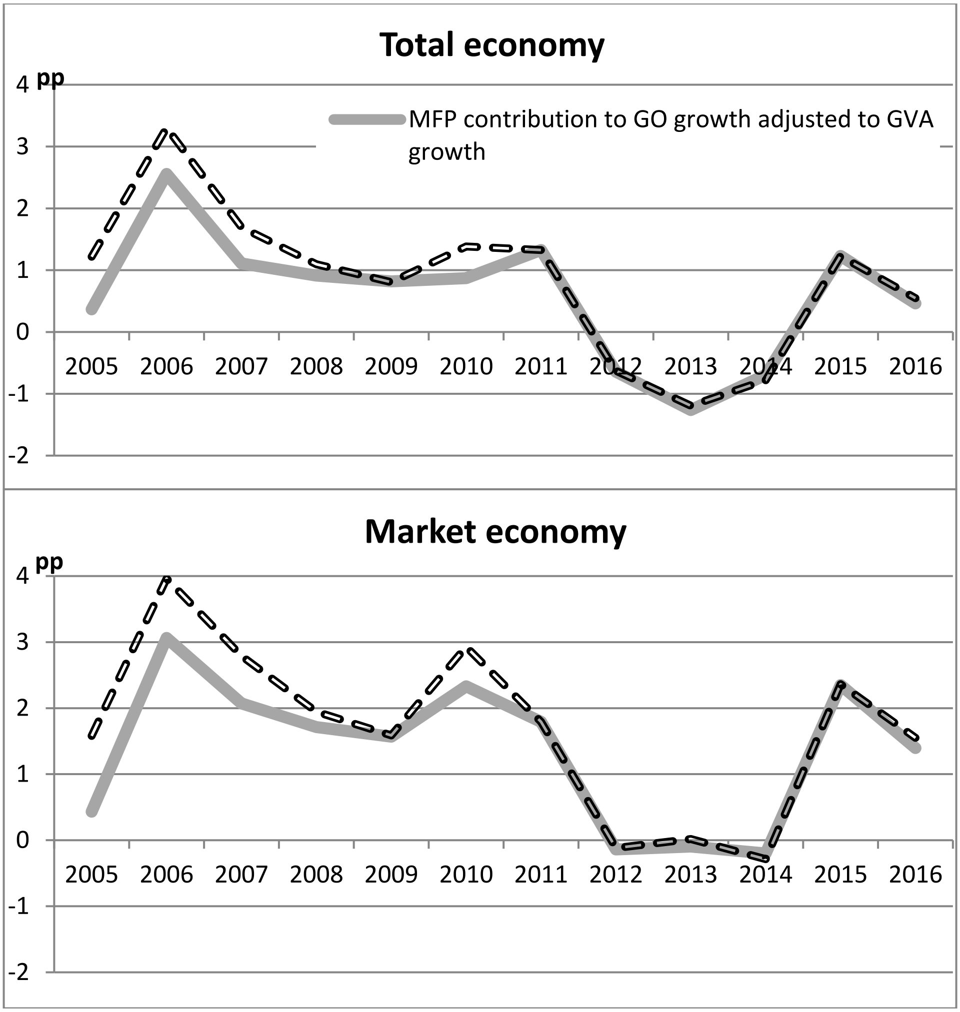
The most essential issue, however, is that the above-mentioned processes can be traced within the framework of KLEMS growth accounting, both at the aggregate and industry levels. If the quality of data on intermediate consumption and on gross output is satisfactory, and the mathematical tool effects associated with the necessity of performing additional calculations are negligible, then the additional computations associated with gross output growth decomposition can be beneficial for the economic analysis. They allow the monitoring of outsourcing in the economy from the perspective of the aggregate economy. Within the framework of KLEMS growth accounting, this can also be done at the industry level. Finally, these

[13] Or that there are no changes in vertical integration impacting MFP growth. This analysis is consistent with the analysis carried out by Gu (2016, pp. 10–11).

[14] The question which factors can be considered as primary is not being answered here. The authors follow the approach presented e.g. by Hulten (2009).

processes could be observed in even greater detail if intermediate inputs were divided into three categories, i.e. energy, materials and services.[15]

Figure 1. MFP contribution to GVA growth calculated straightforward (value-added based) compared to MFP contribution to GO growth (gross-output-based) adjusted to GVA growth for the aggregate and market economies (in percentage points)



Note. Market economy is defined in a standard way as the total economy without NACE sections L, O, P and Q.
Source: authors' work based on Główny Urząd Statystyczny (GUS, 2019).

The comparison of the two values for the MFP contribution from Equation (7) is informative, as shown in Figure 1. It allows the observation of the evolution of the

[15] The research associated with this potential subdivision is under way in Statistics Poland.

above-mentioned substitution processes over time. On the basis of Figure 1, it can be asserted that from 2011 onward, the process of the substitution of production factors (labour services and capital services) by intermediate inputs has gradually ceased, which raises the question why it has been so. The fact that the year 2009 stands out as an exception can be associated with the Financial Crisis shock (2007–2009) that temporarily stopped the change (i.e. transition) processes (which started to slow down already in 2008, as shown on both graphs in Figure 1). Thus, to some extent, it can be considered as an additional confirmation of the validity of the calculus and its underlying methodological content, because it anchors the studied phenomenon to a known empiric situation. The fact that this issue can be interpreted in a similar way for the category of the market economy (as seen on the lower graph in Figure 1), reinforces the likelihood of these findings, and additionally suggests that this phenomenon is not generated by the industries controlled and supported by the central government.

Moreover, in the case of a transition economy such as Poland, the outsourcing expansion, thought as the major component of the substitution process described earlier, can be considered a litmus test (sensor device) for the ongoing changes towards a mature market economy. This is because the Polish pre-transition economy consisted of huge state-owned companies to a much greater extent than nowadays. They had to be 'unbundled', divided and sold to the private sector, which led to the reduction of vertical integration between firms and, subsequently, to the 'unveiling' of outsourced activities between the formerly integrated firms. Moreover, the free-market forces afterwards forced these unbundled, divided and privatised firms to further outsource some of their activities, this time without the public intervention. Bringing this outsourcing expansion (from a macroeconomic (aggregate) perspective) to an end in 2011 meant that the two processes, i.e. the privatisation with unbundling and the free-market expansion of externally provided services ceased to take place. This, however, has to be understood as reaching an equilibrium between two converse processes, i.e. outsourcing and vertical integration. As such, this is consistent with the basic growth theory.[16]

The fact that the substitution process of the contributions of production factor services by the contribution of intermediate inputs might contain more content than only the outsourcing, reinforces the earlier assertion about the litmus test. It seems that the Polish economy has achieved some degree of maturity as a market economy, and in 2011 the country entered a stability phase. There seems to be no other

[16] The growth theory based on the initial growth model of Solow (1956) is a market-equilibrium-based theory. The basic Solow's decomposition (1957), being the predecessor of KLEMS decompositions, is rooted in this theory.

plausible explanation for this phenomenon (the change in the above-mentioned substitution). Therefore, we can continue the analysis by looking at separate industries, which, if orderly, can reinforce this conclusion even more.

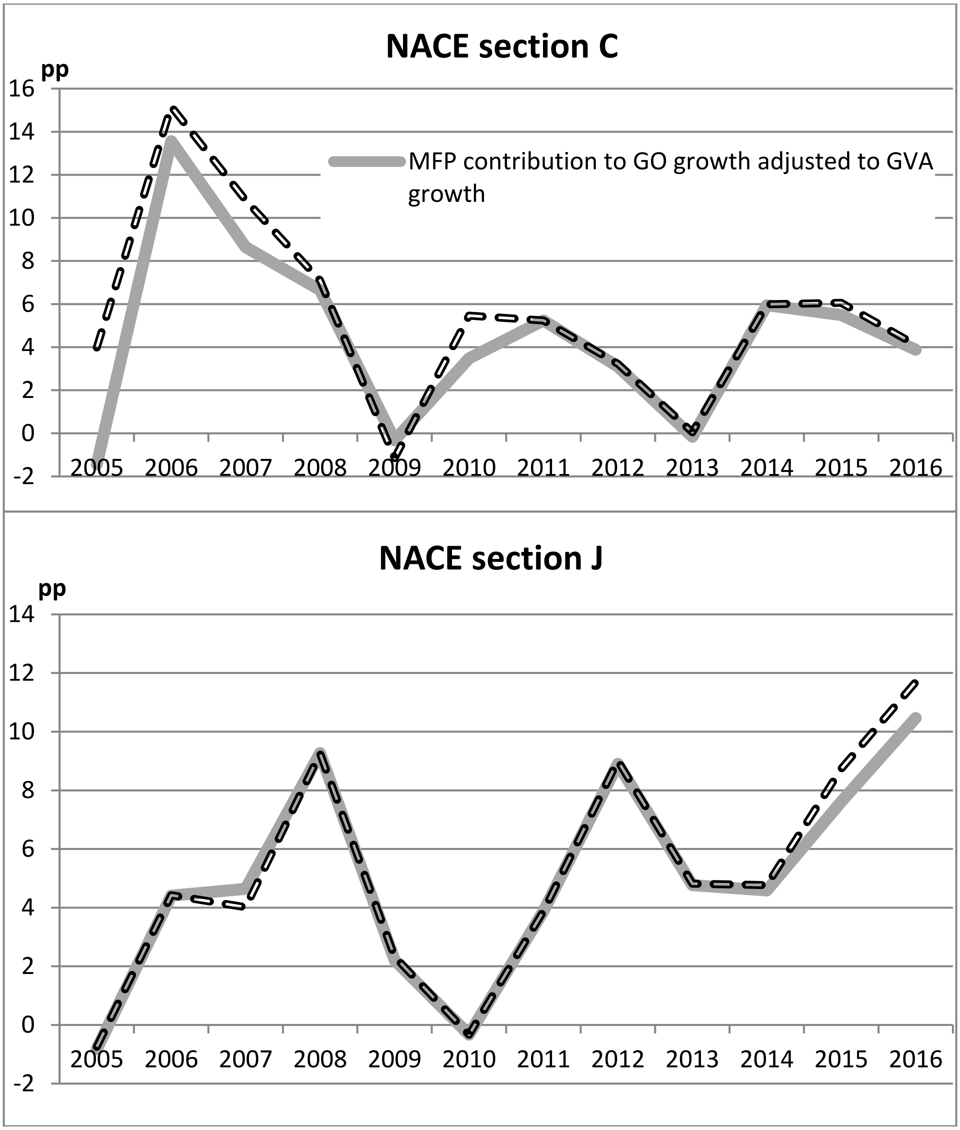
4. Sectoral analysis

In the analysis carried out by industries at NACE 2 (Nomenclature statistique des Activités économiques dans la Communauté Européenne) sections level (European equivalent of Standard international trade classification (SITC 4)), what is worth acknowledging is the fact that the contribution to the above-mentioned substitution of primary production factors by intermediate consumption is originating almost entirely in the C section of NACE 2, i.e. in the manufacturing group of industries, which can be considered a 'heavy weight' NACE section, accounting for almost a quarter of the Polish economy (which is not surprising, though, as manufacturing plays an important role in many economies). We can see it happening on the upper graph on Figure 2. Similarly to the entire economy, the substitution between the factors and intermediate consumption in manufacturing is observed until the year 2011 (with a break between 2008 and 2009), and disappears afterwards.

Another section of NACE rev. 2 which is of interest for this analysis is section J, consisting of industries related to ICT (information and communications technology) industries.[17] It can be seen that the change caused by the expansion of outsourcing, understood as the main medium of substitution, concerns mainly the two last years covered by the analysis, i.e. 2015 and 2016. This suggests that the structural change within the ICT section has only started to get deployed. Therefore, either a specific delay for the Polish economy in the deployment of ICT industries is observed, or a more general, worldwide change is just showing its first effects in Poland.

[17] Appendix I provides graphs for all NACE sections.

Figure 2. MFP contribution to GVA growth calculated straightforward (value-added-based) compared to MFP contribution to GO growth (gross-output-based) adjusted to GVA growth for NACE rev. 2 sections C and J (in percentage points)



Source: authors' work based on GUS (2019).

The analyses for other NACE 2 sections are much less informative. Usually, the process of substitution is inexistent or relatively insignificant there. In sections B and D–E, the economic policy decisions of the central government impacting the vertical integration within those industries are responsible for the small scale of the substitution between the primary factors and intermediate inputs. In section F, before the year 2008, a small-scale vertical integration is ongoing, i.e. the process that is converse to the process of outsourcing. Some small substitution change is observed for the H and I sections of NACE 2, for the latter of which (accommodation and gastronomy) some vertical integration is observed in the years 2015–2016. Some minor changes are observed in sections O, P, Q and R–S, mainly associated with sovereign policy.

Given these findings, and for the sake of carrying out this research exhaustively, we performed gross-output-based and gross-value-added-based MFP comparisons at NACE rev. 2 division level for sections C and J. However, the results are greatly distributed with mathematical tools and outlier effects, which accumulate to such an extent that they become visible. Their common feature is, however, that the discrepancies between the two kinds of MFP disappear almost entirely after the year 2011, and they are minor, which confirms to some extent the validity of the calculus that was performed.

5. Conclusions

The estimation of the level of MFP can be performed in two primary ways. One is based on the decomposition of gross-value-added growth, and the other on the decomposition of gross output growth. It is a well-known fact, and there has been substantial discussion going on about which method is better. The gross-value-added-based MFP seems to be more fit for international comparisons, since the differences in vertical integration between countries have no significant impact on it. The gross-output-based MFP is free from the substitution impact between the production factors and intermediate consumption. So, if additional computations related to the gross-output-based MFP are not conducive to any substantial mathematical tool effects, and data on intermediate consumption are readily available and of good quality, the possibility of converting it into the MFP contribution to gross value added (instead of gross output) seems to be solving the theoretical issue of gross-value-added-based MFP being impacted by the above-mentioned substitution. It is so because the gross-output-based MFP is considered the correct one according to the adopted theory.

However, this issue can be viewed from another perspective. The two kinds of MFP can be considered as equally valid, but of a slightly different essence. If so, they can be both used in economic analyses related to observing the change in vertical integration in the economy – vertical integration being the process opposite to outsourcing. If there is no substantial change in the level of vertical integration in a given aggregation, the difference between the values yielded by the two kinds of MFP appears negligible at that aggregation level, and can therefore be treated as a 'litmus test' for either the expansion or contracting of outsourcing. Since it seems reasonable to assume that the change in the level of outsourcing is strongly related to structural or transitional changes in economies, it can be used for monitoring whether a given economy is undergoing these major changes, or has already moved beyond them. This issue is also relating to tax regulations concerning the business sector, therefore it seems advisable for economic policies not to interfere in such a way as to disturb the market equilibrium between outsourcing and vertical integration.

In the case of Poland, the 'sensor device' based on the two kinds of MFP can be used to assess whether the economy has matured to the level of a standard market economy, and to observe some new developments in this regard. In the light of the KLEMS growth accounting recently implemented in Poland, it seems that most of the changes associated with the transition to the market economy finished in 2011, as far as outsourcing and the related issues are considered. It is also confirmed at the level of industry aggregations, since it concerns mainly manufacturing represented by NACE rev. 2 section C, which underwent a particularly deep transition in Poland. One notable exception is NACE rev. 2 section J, associated with information and communication technology (ICT-related group of industries), where this sort of changes has just begun to accelerate.

The methodology developed in this paper for the purpose of regular computations is novel, although based on known and well-explained processes. It seems capable of being successfully applied to studying other economies for which data necessary to compute the two kinds of MFP are available.

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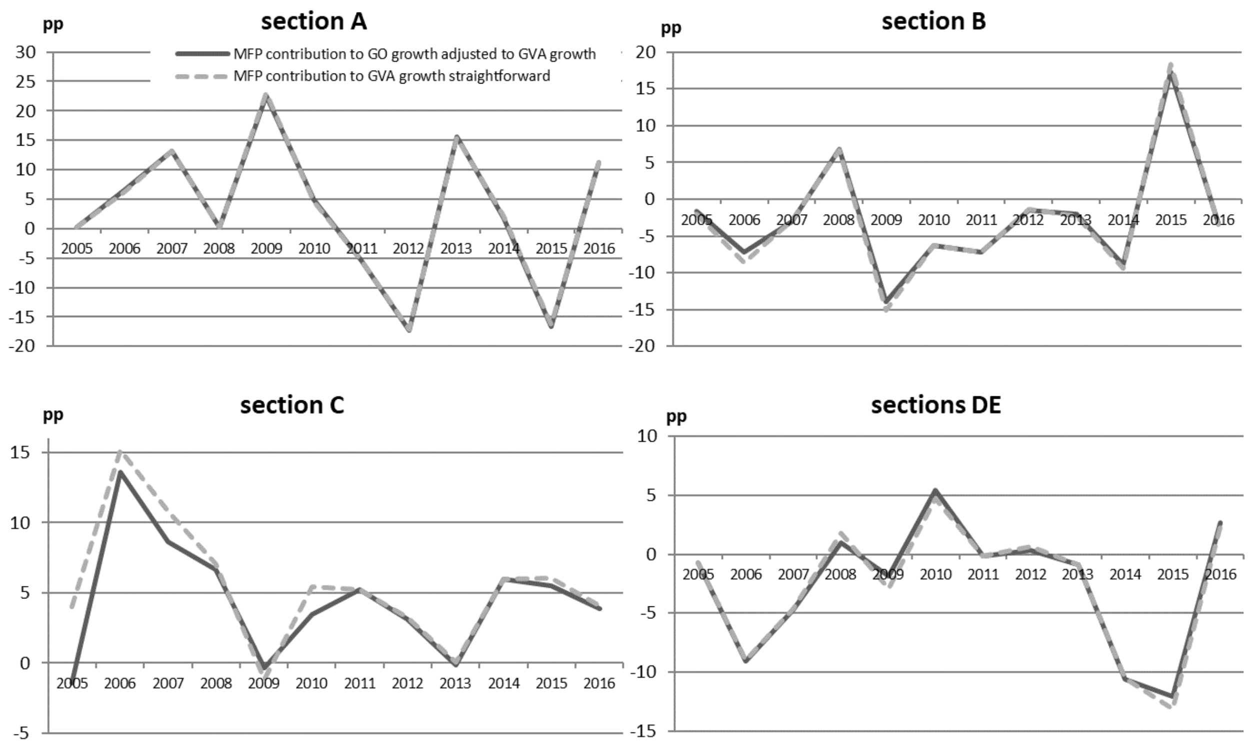
APPENDIX (I)

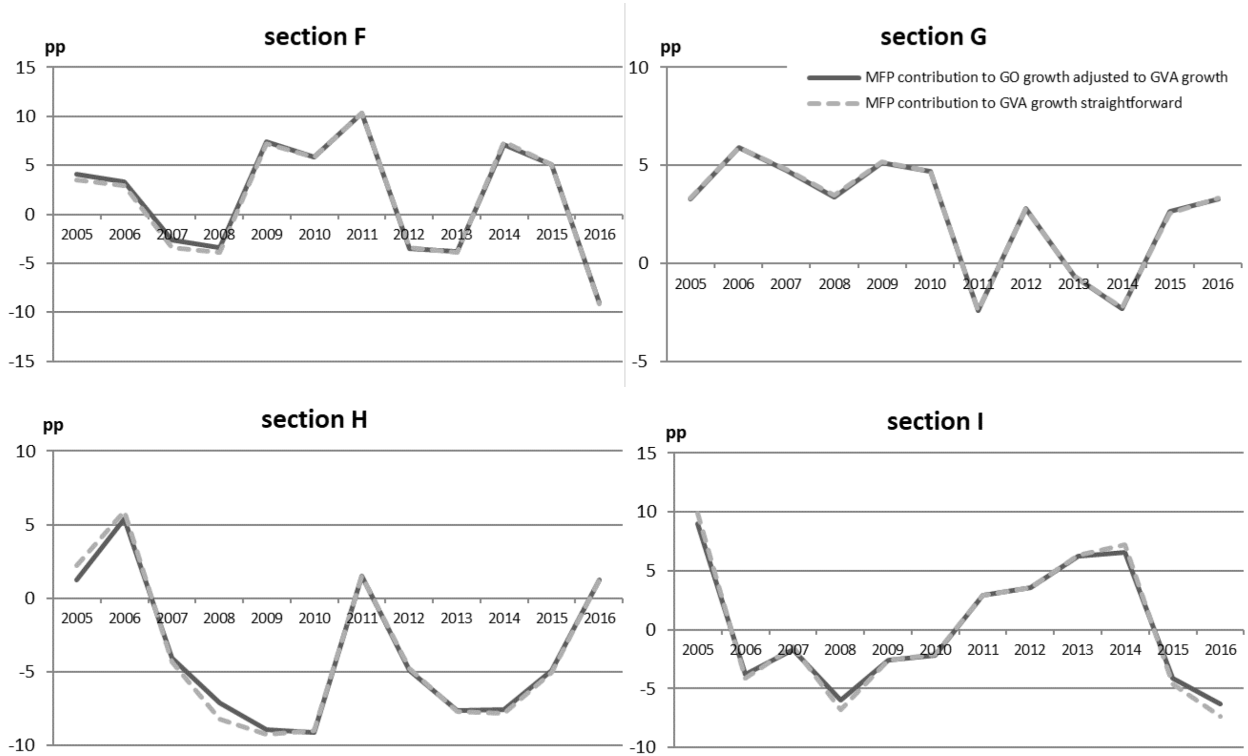
MFP contribution to GVA growth calculated straightforward (value-added-based) compared to MFP contribution to GO growth (gross-output-based) adjusted to GVA growth for NACE rev. 2 sections A, B, C and D-E (in percentage points)



Source: authors' work based on GUS (2019).

APPENDIX (II)

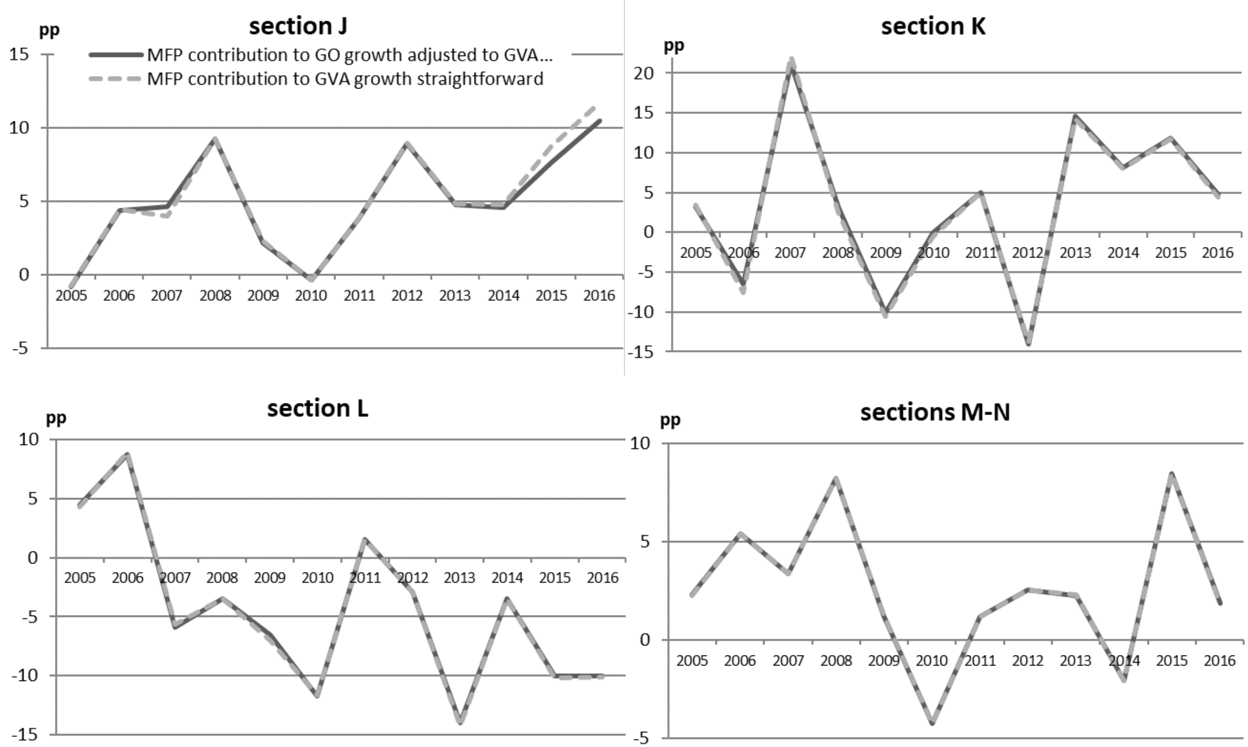
MFP contribution to GVA growth calculated straightforward (value-added-based) compared to MFP contribution to GO growth (gross-output-based) adjusted to GVA growth for NACE rev. 2 sections F, G, H and I (in percentage points)



Source: authors' work based on GUS (2019).

APPENDIX (III)

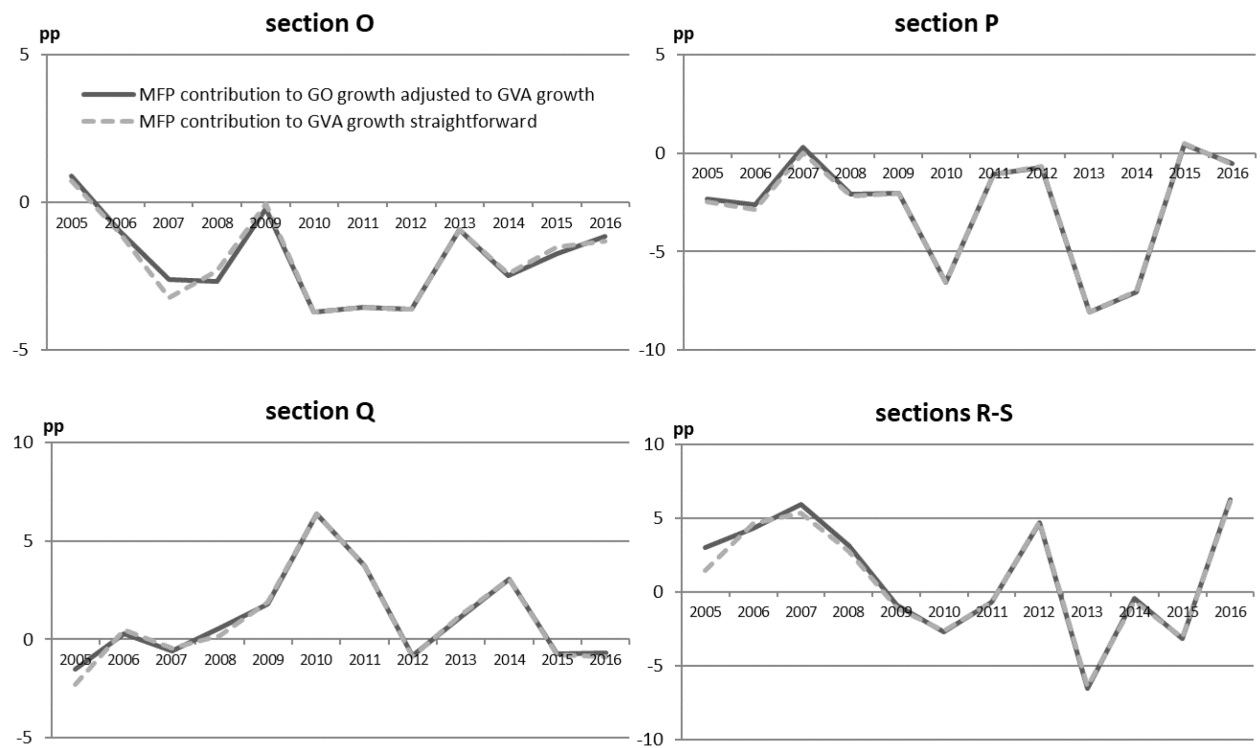
MFP contribution to GVA growth calculated straightforward (value-added-based) compared to MFP contribution to GO growth (gross-output-based) adjusted to GVA growth for NACE rev. 2 sections J, K, L and M-N (in percentage points)



Source: authors' work based on GUS (2019).

APPENDIX (IV)

MFP contribution to GVA growth calculated straightforward (value-added-based) compared to MFP contribution to GO growth (gross-output-based) adjusted to GVA growth for NACE rev. 2 sections O, P, Q and R-S (in percentage points)



Source: authors' work based on GUS (2019).

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How the shadow economy can be detected in National Accounts[1]

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Abstract. The paper examines how indicators of the shadow economy correspond to the National Accounts values. More precisely, we focus on household accounts assuming that the shadow economy should be visible in the difference between household income and consumption, as household (disposable) income is grossly underreported. Household consumption seems therefore to be a more accurate indicator in this context, as most shadow economy income is eventually spent on consumption. This implies that household savings figures should be negatively related to the values of the shadow economy; consequently, if the values relating to the shadow economy are high, savings should be low, or even negative, and vice versa. We verify this hypothesis using European cross-country data covering the years 1991–2017 with the application of MIMIC model calculations as a point of reference. The estimation results lend very little support to the hypothesis assuming that the shadow economy depresses household savings, even though we can otherwise explain comparatively well the cross-country variation in household savings and consumption growth rates.

Keywords: shadow economy, National Accounts, saving behaviour

JEL: C390, C510, C820, H110, U170

1. Introduction

Literature on the shadow economy presents numerous methods of measuring the volume of this kind of economy, which is not surprising, as measuring it in the same way as other economic phenomena is difficult. Thus, most methods are indirect to some degree, as seen in the extensive survey of e.g. Kirchgässner (2017) or United Nations Economic Commission for Europe (UNECE, 2008). To sum up, there are various survey studies, studies using payment media data (e.g. Takala & Virén, 2010), employment data or discrepancies in national accounts, as well as analyses dealing with tax receipts (tax gap) and different model-based analyses. In this latter category, the most popular set-up involves the MIMIC model approach, propagated by Friedrich Schneider in particular (see e.g. Medina & Schneider, 2019; Schneider & Buehn, 2016). In this model the unobservable (latent) shadow economy variable is modelled by observable forcing variables, using the model restrictions of the (presumed) theoretical model (for details see Schneider & Buehn, 2016).

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The concept of the 'shadow economy' in relation to National Accounts is used as an aggregate of all the economic activities which are missing from National Accounts. Thus, they consist of what is referred to as the grey economy (mainly tax evasion), the illegal economy and unreported income. According to National Accounts, household production is not considered a part of the shadow economy.

When analysing the shadow economy, we use the estimates of Medina and Schneider (2019) as a point of reference, as they are by far the values most widely published and referred to, and because they relate to practically all the countries in the world.[2] The aim is to see how these values correspond to the official National Accounts measures. Our hypothesis assumes that if these estimates are 'correct', then some traces of the implied values of the shadow economy should also be visible in the National Accounts. The basic idea is then that the shadow economy appears disproportionally in different National Accounts measures. As is well known, all transactions in the National Accounts are shown in production, income and in the use of income/production accounts. There is convincing evidence that it is the income measures that distort the shadow economy more than other National Accounts measures. As a result, the total income in most cases is likely to exceed the total use of income (i.e. the sum of the demand components). This way of measuring the scope of the shadow economy is mentioned in almost all literature surveys (e.g. Gyomai & van de Ven, 2014; UNECE, 2008) and yet relatively little serious effort has been put thus far to examining whether the idea can be applied to actual data.[3]

It seems, however, that a proper analysis cannot be done at the level of Gross National Product (GDP) nor Gross National Income (GNI), as many income transfer components and consolidations of income between different (sub)sectors (including the rest of the world) and industries are involved; additionally, discrepancies are often considered as an indication of the low quality of a statistical compilation.[4] Therefore, this paper will concentrate on one sector only – households. In this case, the income and expenditure approaches produce (by definition) different outcomes and thus statisticians have no incentive to manipulate the

[2] There are some other interesting data sets like Elgin's (2020) data on European metropolises, but they do not facilitate considerably the comparison with the data of the National Accounts.

[3] There are also some other pitfalls in the measuring of the shadow economy, e.g. a part of the shadow income could be transferred abroad. Nevertheless, most of it would likely be done via the banking system and therefore would show in the current account and further in income accounts. Some income could be hoarded but that would probably be a temporary behavioral pattern not lasting for as long as 27 years, which is the period that the data used here covers.

[4] Schneider and Buehn (2016) argue that since national accounts statisticians are anxious to minimise this discrepancy, the initial discrepancy or the first estimate should be employed as an estimate of the shadow economy rather than the published discrepancy. If all the components on the expenditure side were to be measured without error, this approach would indeed yield a good estimate of the size of the shadow economy. Unfortunately, this is not the case. Instead, the discrepancy reflects all the omissions and errors in the national accounts statistics and the shadow economy. These estimates may therefore be crude and of a questionable reliability.

discrepancies between these two. Household consumption and income are also frequently surveyed for different statistical purposes (such as income distribution indicators and the consumer price index). In practical terms, we will be comparing household income and household consumption. The basic idea is that there is a shadow income component in household (disposable) income that is not included in the National Accounts values of household income. As regards household consumption, there can also be a shadow income component, but we believe that this component is much smaller than the corresponding income component, as all income is either consumed or invested over time. Income from the shadow economy is consumed much in the same way as the income from the non-shadow economy. In fact, this idea is often utilised in practical anti-corruption and anti-tax-evasion procedures in a very simple way: individuals' consumption level is compared with their official income. In practice this entails surveying the housing space, the number and price of cars, etc.[5] Of course, the real household consumption includes some items that are not present in the National Accounts statistics. Most notably this is true in the case of such 'illegal' components as prostitution and drugs. Although the volume of these components varies both across countries and over time, the average value might still be rather low and not significant from the point of view of our empirical results.[6]

The problem here is that at the theoretical and behavioural level, we do not have the identity of $consumption = income$ that would hold every period, nor do we have a simple degenerated equation for consumption being equal to $b * income$, where the propensity to consume b would be universally constant over time and households/countries. Nevertheless, it could temporarily be assumed that the relationship between $income$ and $consumption$ – at least in the long run – would be relatively constant. Then, other things being equal, we could expect that in households, and thus in countries where the shadow economy reaches high levels, the share of b tends to be large. In fact, b could be well above 1 and, consequently, the savings rate would be negative. Therefore, we intend to scrutinise the correspondence between the (long-run) measures of the shadow economy and the level of the savings rate. We attempt to answer the question whether the savings rate is small or negative in countries with a large shadow economy, and if the opposite is true for economies with a small shadow economy. Alternatively, we will focus on the dependence of (the growth of) consumption on the measures of the shadow economy. As far as the consumption growth is concerned, we expect its positive dependence on the size of the shadow economy, conditional on the measured National Accounts income growth and other control variables.

[5] See Enikolopov and Mityakov (2019) for a practical research application.
[6] For instance, Statistics Finland's estimate of the size of these items is only 0.2% of the Finnish GDP.

Our approach is to some extent related to an old study by Pissarides and Weber (1989), where the household (food) consumption – income relationship is analysed from the point of view of the grey economy. Pissarides and Weber use the UK Family Expenditure Survey data to find out whether the self-employed underreport their income. The authors adopt some comparatively strong assumptions on the permanent income consumption model on the basis of which they develop an equation where the measured income and the indicator of self-employment (jointly with a set of controls for household characteristics) appear on the right-hand side of the equation. The estimation results indicate that a substantial underreporting of income is indeed related to self-employment. A more comprehensive study was performed by Lyssiotou et al. (2004), who based it on an expenditure system of six main (non-durable) commodity groups and information on the main sources of income. From our perspective, the interesting point in these studies is the assumption that consumption expenditure – unlike disposable income – is assumed to be correctly measured (see Adair (2018) for some critical comments on other features of this study).

In the subsequent parts of the paper, we review both the Medina and Schneider (2019) and the National Accounts data or the data for different controls. Then, in Section 3 we present the estimates using cross-country panel data, while Section 4 contains the concluding remarks.

2. The data

We begin with an analysis of the shadow economy data. In the Medina and Schneider (2019) study, there are 158 countries and in the majority of the cases the data cover the period 1991–2017, whereas here we consider 34 European countries. Most of them are European Union (EU) countries, but the sample also includes Iceland, Norway, Switzerland, the United Kingdom, Ukraine, Belarus and Russia. Mexico and Colombia (Organisation for Economic Co-operation and Development [OECD] countries) are also included into the sample to examine the dynamics of the results.

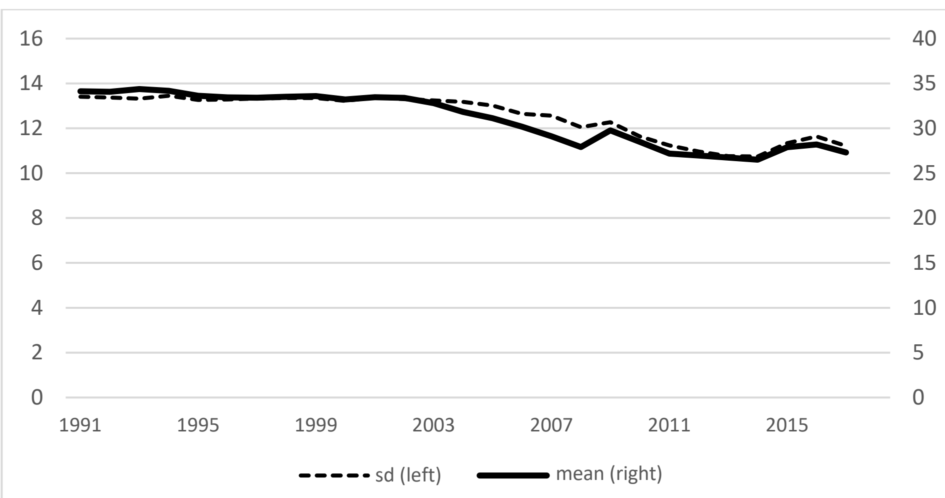
As regards the National Accounts data, the key variables are private consumption, household disposable income, and the savings rate. We consider both gross and net income (and, accordingly, gross and net savings), but since the measures do not make any noticeable difference in the results, we concentrate on the net values. The coefficient of correlation between the two series is 0.97, which mainly reflects the level differences. The details of the data are explained in the data appendix.

As for the controls, we have GDP (both in national currencies and in US dollars), the respective deflators (including the consumption prices), the (real) income *per capita* in euros and real GDP *per capita* in US dollars, the share of agriculture, the share of self-employment, the real (long-term) interest rate and, finally, the amount of remittances sent to and from the country. We use the total population numbers for scaling purposes.

Although the control variables could cover longer periods, we decided to restrict the sample to the same years as the Medina and Schneider shadow economy sample, i.e. to 1991–2017. Altogether we could have 972 data points, but the final sample is smaller because of the differencing and lags and since the savings/income data cover a shorter period of time (1995–2017). For these reasons, the final sample size consists of about 600 data points.

Before proceeding to the proper analysis, let us briefly examine the Medina and Schneider (2019) shadow economy data. Some of the typical features of the data are presented in Figure 1, which shows the cross-section means and standard deviations of 158 time series.

Figure 1. Cross-section means and standard deviations of the shadow economy series, expressed in %



Note. sd – standard deviation. The values have been computed from all 158 series.
Source: authors' work based on Medina and Schneider (2019).

The above indicates that the size and country dispersion of the shadow economy has decreased over time. The 2008/2009 financial crisis is shown in the mean values as a small, temporary peak, but otherwise it is difficult to find any cyclical features in the data. This is also reflected in the autocorrelation function of the shadow economy series, where the AR(1) coefficient is 0.955. For the sake of comparison, the corresponding values of the net savings rate and the growth rate of (real) private consumption expenditure are 0.871 and 0.221, respectively.[7]

The trend-like features of the shadow economy time series also appear in a principal component (PC) analysis. Thus, if the analysis is based on the whole of the data, including 158 countries, the first PC explains 76% of the total variation of the data, 3 PCs – 89% and 10 PCs – 97% of the data. It would take 20 PCs to explain 100% (in practical terms) of the total variation. In other words, the role of country-specific features in the data is relatively small, which makes the identification of the shadow economy in the panel data more difficult. This similarity clearly reflects the way in which the data are constructed (the same model, the same forcing variables and similar trends in these variables across countries).

3. Analysis

3.1. Derivation of the model and the hypotheses

The estimating equation for the savings rate takes the following form:

$$s_{it} = \alpha_{0it} + \alpha_1 s_{it-1} + \alpha_2 h_{it} + \alpha_3 \Delta y_{it} + \alpha_4 \pi_{it} + \alpha' \mathbf{X}_{it} + \mu_{it}, \qquad (1)$$

where s denotes the savings rate, h is the shadow economy measure, Δy signifies the growth of real income, π indicates the rate of inflation, $\mathbf{X}$ represents the set (vector) of the control variables, and μ is the error term. Subscript i denotes country, and t time (year). All variables are expressed in real terms, meaning that if they were originally nominal, their values would be deflated by consumer prices. Thus, e.g. $\Delta y = \Delta log(Y/P)$. At this stage, the basic hypothesis is that α_2 is negative, therefore an increase in the shadow economy appears in a larger negative difference between National Accounts measures of income and consumption.

In the same way, we specify the equation for consumption growth as

$$\Delta c_{it} = \beta_{0it} + \beta_1 \Delta c_{it-1} + \beta_2 h_{it} + \beta_3 \Delta y_{it} + \beta_4 \pi_{it} + \beta' \mathbf{X}_{it} + \mu_{it}, \qquad (2)$$

where Δc denotes the growth rate (log difference) of real private consumption growth. Real income refers to the real current income here; following e.g. Pissarides and Weber (1989), we can assume that current income Y is related to permanent income Y^P by the expression $Y = \rho Y^P$, where ρ is a random variable which depends

[7] Some recent comparative analyses on the shadow economy size estimates are reported in e.g. Almenar et al. (2019) and Dybka et al. (2019).

on certain aggregate events. Since we cannot really identify ρ for the shadow and non-shadow economy, we refer to the current income only. Moreover, we cannot see that the shadow economy would affect current and permanent income genuinely differently. As far as Equation (2) is concerned, the basic hypothesis is that β_2 is positive, i.e. an increase in the shadow economy share facilitates higher consumption, given that the National Accounts' measure of real disposable income is the control variable.

When we introduce income growth variable Δy into these two equations, we must assume that there is a difference between the 'true' income and the measured income. Suppose that the true income is Y^*, while the measured income is Y. As regards consumption, however, true consumption C^* is supposed to equal the National Accounts measure of C. Ratio h is assumed to be the share of the shadow economy in the measured income (although it is not completely clear how the Medina and Schneider values should be interpreted). As a result, h is now $\frac{(Y^*-Y)}{Y} = \frac{Y^*}{Y} - 1$, and thus $Y^* = (1+h)Y$. In our estimating equations, we have the (real) income growth on the right-hand side of the equation, but ideally it should read $\Delta log(Y^*)$. Using the previous notation, $\Delta log(Y^*)$ equals $\Delta log(1+h) +$ $+ \Delta log(Y) \approx \Delta h + \Delta y$. Thus, instead of using (the level of) h as the right-hand side variable, we should use its difference (but the signs of both Δh and Δy should be positive). That is clearly true only if we assume that the shadow economy share affects the economy solely via the income variable.

We have an additional problem with the savings rate equation due to the fact that the savings rate also contains a measurement error. The 'correct' savings rate would be $(Y^* - C)/Y^*$ instead of $(Y - C)/Y$. Moreover, the savings rate is highly persistent, as pointed out above, so that the AR(1) coefficient of the lagged value of the savings rate is close to 0.9, which is also visible in the subsequent empirical results. To simplify the matter, let us assume that the left-hand side variable is Δs instead of s (in fact, Deaton (1977) uses Δs as the dependent variable). Then the skeleton form of the savings rate equation, where (a difference in) the savings rate depends only on real income growth, can be written as $\frac{\Delta((1+h)Y-C)}{(1+h)Y} = \beta\Delta h + \beta\Delta y$, where β is the coefficient of $\Delta log(Y^*/P)$ in the savings rate equation. Now the left-hand side of this equation is simply $\Delta(1-\{(1-s)/(1+h)\})$, which may be approximated by $(\Delta h + \Delta s)/(1+h)$.[8] Thus, the equation takes the form of $\Delta s = (\beta(1+h)-1)\Delta h + \beta(1+h)\Delta y$, implying that the share of the shadow

[8] $\Delta(1-(1-s)/(1+h))$ is equal to $\frac{(\Delta s+\Delta h)}{(1+h)} - \Delta h(h+s)/(1+h)^2$ and we disregard the latter term.

economy has a negative effect on the change of the savings rate at reasonable values of h and β, while the National Accounts income growth still has a positive effect. Therefore, in fact, estimating Equation (1) would take the following form:

$$\Delta s_{it} = \alpha_{0it} + \alpha_1 s_{it-1} + \alpha_2 \Delta h_{it} + \alpha_{31} h_{it} \Delta y_{it} + \alpha_{32} \Delta y_{it} + \alpha_4 \pi_{it} + \alpha' \mathbf{X}_{it} + \mu_{it}. \quad (1)$$

Now we would expect the sign of α_2 to be negative and the sign of both α_{31} and α_{32} to be positive. In the empirical application, however, it is difficult to obtain precise estimates for α_{31} and α_{32}; we must therefore rely more on specification (1). Regardless, we use either the level of h or the first difference of h as the dependent variable.

As regards other control variables, we use the rate of inflation and the real interest rate. Applying the rate of inflation as a control variable can be motivated by the Deaton (1977) savings equation, where the inflation rate affects savings due to the following mismeasurement effect: when inflation grows, consumers (sampling individual prices) interpret increases of individual prices as changes of the relative prices of respective commodities and decrease the demand for those commodities. When we aggregate consumers and households, a positive relationship occurs between the savings rate and inflation. Obviously, we would expect inflation to have an inverse effect on consumption growth. That is because (roughly) $\Delta s = \Delta y - \Delta c$.

In addition to the income growth and the share of the shadow economy, we have some other structural variables: the share of self-employment (emp), the share of agriculture (agr), the *per capita* income level ($gdpc$; GDP *per capita* in constant US dollars), the growth rate of population (Δpop), the rate of inflation (inf), the real interest rate (rr) and the amount of remittance income – both inflow (rem) and outflow (rex). The remittance income variables are expressed in US dollars, so they are divided by the respective GDP in US dollars. Household indebtedness ($debt$) is another variable, but not included in the final specification, since not all data were available for each of the studied countries and thus its explanatory power was rather low. A detailed list of the variables and data sources as well as their descriptive statistics are included in the Appendix.

3.2. Empirical results

We start by reporting a set of cross-section results for sample means of the main variables. These are presented in Table 1.

Table 1. Estimation results with mean values of the country data

Dependent Variable→	$Shadow$	sn	Δc
Constant	19.747 (36.16)	2.868 (3.11)	0.028 (11.74)
$self\text{-}emp$	0.137 (5.48)	0.290 (12.57)	–0.046 (7.79)
$gdp\ pc$	–0.018 (20.02)	0.049 (4.56)	–0.025 (091)
rem	0.291 (7.13)	.	0.555 (5.43)
rex	.	–0.017 (6.71)	.
agr	–0.031 (0.47)	1.022 (25.99)	0.096 (9.28)
inf	0.816 (20.45)	0.735 (17.63)	–0.073 (6.50)
sn	0.018 (0.49)	.	.
$shadow$	.	0.046 (1.36)	–0.005 (5.02)
R^2	0.818	0.570	0.275
SEE	4.199	4.523	0.012

Note. Number of observations: 36. The numbers in parentheses are the t-values. $self\text{-}emp$ – self-employment, $gdp\ pc$ – GDP *per capita*, rem – remittance income inflow, rex – remittance income outflow, agr – agriculture, inf – rate of inflation, sn – net saving rate, $shadow$ – shadow economy measure, Δc – growth rate (log difference) of real private consumption growth.
Source: authors' calculations.

It is quite clear that the measure of the shadow economy does not seem to be related either to the savings rate or to the growth rate of consumption. On the other hand, the figures demonstrate that the size of the shadow economy is negatively related to the income level of the country and positively related to the level of self-employment and the rate of inflation. These results are not very surprising, as this type of variables drive the MIMIC model predictions for the shadow economy share.

Subsequently, we proceed to the ordinary panel data and estimate equations for the net savings rate and consumption growth. The respective results are presented in Table 2.

Table 2. Panel estimates of consumption and savings rate equations

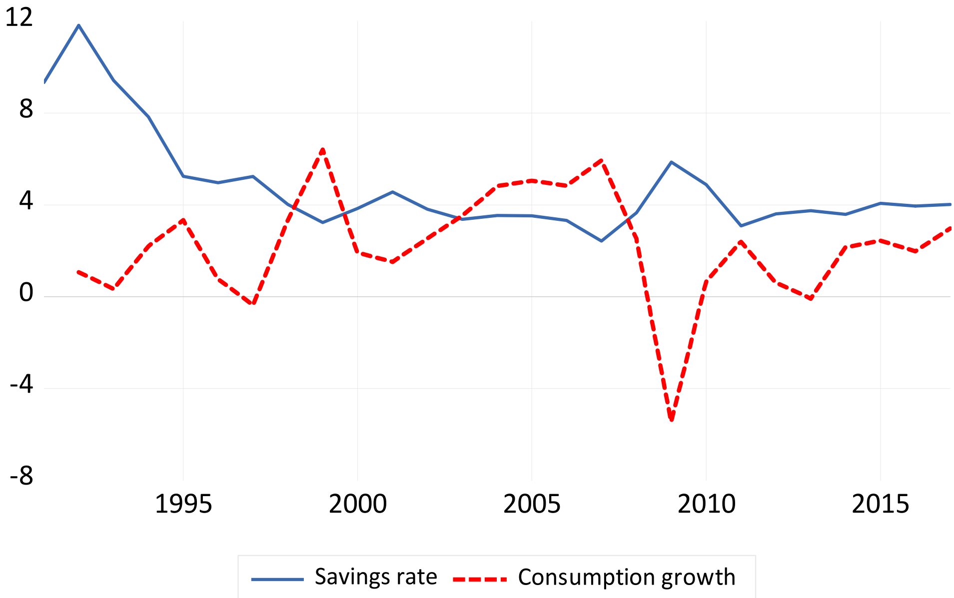
Dependent Variable→	Δc	Δc	Δc	Δc	sn	sn	sn	Δsn	sn
Constant	0.021 (1.79)	0.045 (2.08)	0.015 (2.89)	.	–0.434 (0.64)	4.395 (2.25)	–0.472 (1.08)	–0.616 (1.04)	.
Lag of Dep. Var.	0.159 (2.96)	–0.095 (1.79)	0.189 (3.44)	–0.009 (0.33)	0.866 (45.53)	0.706 (20.34)	0.866 (50.17)	–0.137 (6.31)	0.739 (17.41)
$shadow$	0.001 (0.28)	–0.179 (1.42)	.	.	–0.036 (1.56)	0.142 (1.49)	.	0.422[a)] (1.46)	.
$\Delta(shadow)$	.	.	–0.900 (5.67)	–0.006 (2.45)	.	.	0.747 (6.54)	0.763 (5.32)	0.657 (5.31)
Δy	0.281 (7.30)	0.228 (6.45)	0.427 (6.18)	0.414 (31.77)	0.146 (7.03)	0.149 (6.89)	0.183 (8.75)	0.088 (1.44)	0.228 (18.36)
$inflation$	–0.189 (2.31)	–0.401 (6.34)	–0.214 (2.62)	.	0.217 (3.69)	0.281 (2.66)	0.218 (7.26)	0.248 (4.34)	0.445 (16.07)
$self\text{-}emp$	–0.049 (2.11)	–0.240 (3.22)	–0.059 (2.62)	–0.002 (0.80)	0.014 (0.67)	0.204 (2.66)	0.024 (1.18)	0.025 (1.33)	0.439 (4.91)
agr	0.315 (5.87)	1.198 (7.63)	0.313 (5.86)	0.007 (2.07)	–0.147 (4.33)	–0.430 (3.84)	–0.153 (4.89)	–0.158 (3.63)	–448 (16.07)
rr	–0.384 (5.50)	–0.663 (8.83)	–0.324 (4.74)	–0.477 (16.79)	0.150 (3.90)	0.202 (4.73)	0.092 (2.42)	0.101 (1.71)	0.151 (4.74)
$gdp\ pc$	0.022 (0.31)	–0.296 (1.94)	–0.004 (0.66)	–0.001 (2.56)	0.002 (0.25)	0.002 (1.07)	0.014 (2.19)	0.017 (2.32)	–0.011 (0.65)
Δpop	0.282 (1.41)	–0.450 1.72)	0.467 (1.68)	–1.393 (2.36)	0.267 (1.41)	0.487 (2.11)	0.459 (0.21)	0.282 (0.13)	–0.091 (0.91)
Net remittances/Y	–0.024 (1.91)	0.668 (2.76)	–0.021 (1.77)	–0.020 (1.32)	–0.002 (0.16)	–0.005 (0.96)	–0.003 (0.18)	–0.003 (0.15)	0.002 (0.26)
Fixed effects	no	cs fixed	no	dif	no	cs fixed	no	no	dif
Estimator	OLS	OLS	OLS	GMM	OLS	OLS	OLS	OLS	GMM
R^2	0.551	0.679	0.586	.	0.886	0.900	0.894	0.248	.
SEE	0.0280	0.0250	0.0280	0.0236	2.580	2.496	2.501	2.495	2.244
DW/J	1.50	1.62	1.52	0.243^J	2.03	1.91	1.98	1.98	0.222^J
Observations	584	584	584	536	580	580	580	580	536

Note. The numbers inside the parentheses are robust t-values. $cs\ fixed$ denotes fixed country effects. Superscript J denotes the p-value of the J-test. Dif indicates that the data are differenced. In the second to last column, the dependent variable is differenced. In this column, variable indicated by [a)] is $h * \Delta y$ according to Equation (1′). Given that the sample mean value of the shadow economy is 21%, the elasticities of Δy are roughly the same in Equations (7) and (8) in Table 2. $shadow$ – shadow economy measure, $\Delta(shadow)$ is a difference of it, Δy – growth of real income, inf – rate of inflation, $self\text{-}emp$ – self-employment, agr – agriculture, rr – real interest rate, $gdp\ pc$ – GDP *per capita*, and Δpop – growth rate of population.

Source: authors' calculations.

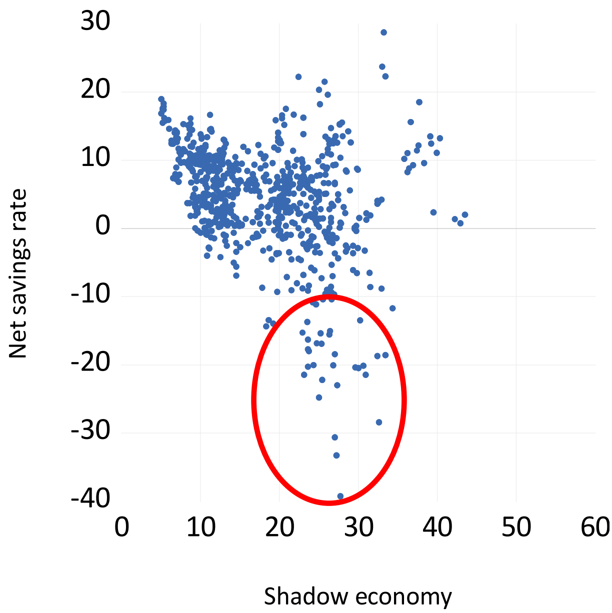
The data for the time series of the savings rate and the consumption growth rate are shown in Figure 2 and the scatter plots for the data of the sn, Δc and the shadow economy measure h are shown in Figures 3 and 4. The scatter plot between the shadow economy measure and household indebtedness is shown in Figure 5. It seems that in the 'shadow economy countries' indebtedness is typically low, but it is difficult to state at this point whether the relationship has any deeper meaning.

Figure 2. Mean savings and consumption growth rates in the data, expressed in %



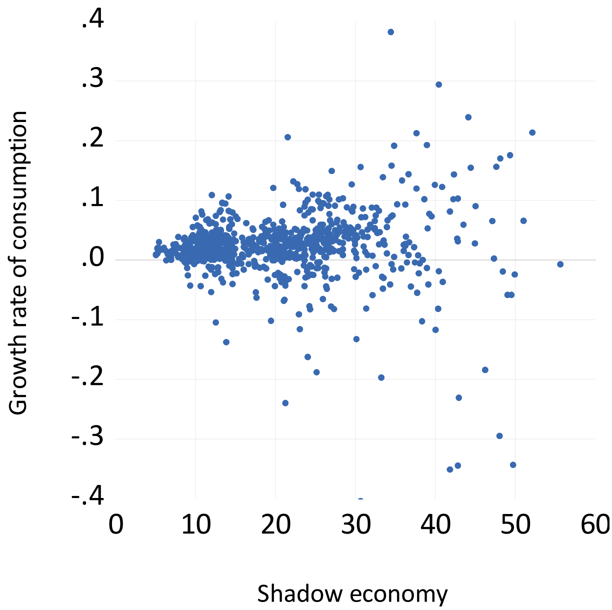
Source: based on authors' calculations.

Figure 3. Shadow economy share and the net savings rate



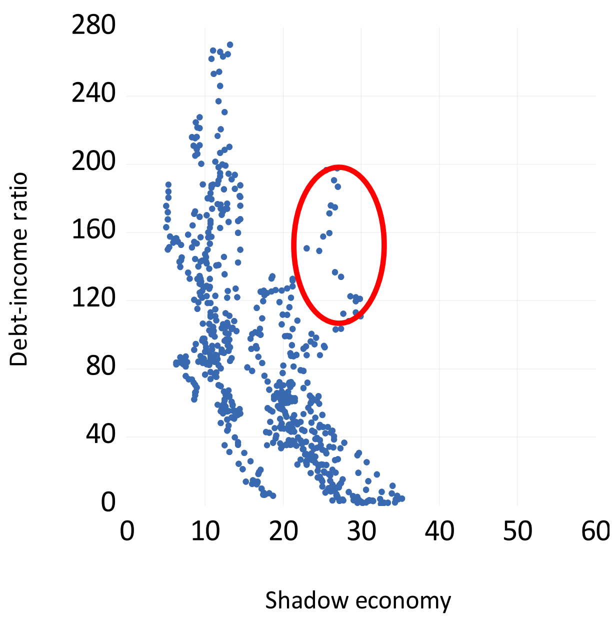
Note. Observations in the circled area come from Greece, Cyprus, Latvia and Romania.
Source: based on authors' calculations.

Figure 4. Shadow economy and consumption growth



Source: based on authors' calculations.

Figure 5. Shadow economy and household indebtedness



Note. The North-East set of observations come from Cyprus.

Source: based on authors' calculations.

When estimating (1) and (2), we face the problem of reverse causality between savings or consumption growth on the one hand, and the shadow economy variable, on the other. In our opinion, however, the nature of shadow economy is such that it is most likely not affected by changes in the savings rate or the growth rate of consumption. Thus, the shadow economy is close to the concept of a 'deep' variable. Even so, when we use the (Arellano-Bond) GMM estimator, we assume that the shadow economy variable is endogenous in estimating Equations (1) and (2). The use of GMM is obviously required also due to the panel setting of the data.

The mean savings rate for the 34 countries is remarkably stable over time, while consumption growth (more) clearly reacts to cyclical variations of real income. As opposed to the shadow economy measures, savings and consumption growth rates show no visible trends. As regards the relationship between the shadow economy on the one hand and savings and consumption on the other, we see from Figure 3 that the savings rate seems to be inversely related to the shadow economy. This, however, results from certain extreme observations: Romania has a very high negative savings rate and a very high value for the shadow economy,[9] while Switzerland is characterised by a very low value of the shadow economy, but a very high savings rate. On the other hand, all other observations do not follow any clearly defined pattern. The very high negative savings rates of some countries are puzzling, which is particularly true for Romania, where the negative rate does not seem to be a temporary phenomenon and it differs from that of its neighbouring countries. When comparing the Romanian savings rate with the household indebtedness variable, some correspondence can be detected as the indebtedness had increased from practically zero to 30% in the sample period; that, however, does not match the magnitude of the cumulative sum of the negative savings rates.[10]

As far as consumption growth is concerned, it is very difficult to distinguish any kind of relationship with respect to the size of the shadow economy. If our basic assumption that income is more distorted by the shadow economy than consumption is true, we might expect a positive relationship between the shadow economy and consumption growth, and yet this kind of pattern does not seem to exist. Obviously, the identification of the shadow economy effect becomes more difficult if the relative size of the shadow economy remains constant.

Now let us refer to the cross-country analysis of the time-series data. As previously mentioned, all results are presented in a conventional panel data setting.

[9] The case of Romania is discussed in more detail in Rocher and Stierle (2015). It is suspected that a part of the country's consumption is in fact investment and the distinction between households and firms is made incorrectly.

[10] Within the whole data set, we found the following correspondence between indebtedness and the savings rate: $\Delta dept = -0.19s$, which is obviously far from the identity of $\Delta dept = s$. However, it must be kept in mind that debt refers to the gross (not net) debt ratio.

This kind of data requires considering the issue of fixed and random effects first, and therefore we ought to treat the fixed effect with caution. This is because the shadow economy variables, as well as most of the control variables, are enormously persistent, coming close to linear trends. If we had fixed country effects, they could absorb most of the impact of the shadow economy. Nonetheless, we do also use the fixed effects specification as an alternative. Regarding the random effects, we found that this specification is not appropriate, as the Hausman test indicated. In the case of the GMM, we use the first differences of the data. When checking the robustness of the results, we also apply robust estimators to eliminate the potential effect of any outliers.

The results in Table 2 confirm the initial impression that the shadow economy measures do not help predicting either the level of savings or the growth rate of consumption. If the level form of the shadow variable is used in the estimating equation, the signs of the coefficients are either 'wrong', or insignificant, according to the standard levels of significance. Alternatively, if we use the first differences of the shadow economy variable, the t-values are very high, but the signs of the coefficients do not make sense from the point of view of the analysed notion assuming that the shadow economy reveals itself in income but not in consumption. Thus, the estimates imply that an increase in the (change of) the size of the shadow economy decreases consumption growth (given income growth and other controls) and, accordingly, an increase in the (change of) the shadow economy increases the National Accounts savings rate. When the equation was estimated by the GMM estimator, the results remained practically unchanged in relation to the key variables. It only involved the change of some of the control variables' (such as the population growth and GDP *per capita*) coefficients along with the estimator and the differencing of the data.[11]

Otherwise, the estimated equations perform reasonably well following the lines of the earlier research. In conclusion, both consumption growth and the savings rate are sensitive to income growth, the real interest rate and the rate of inflation. The savings rate equation works perfectly according to Deaton's (1977) 'involuntary saving hypothesis', i.e. inflation indeed increases savings in the same way as real income growth does. The real interest rate also changes in accordance with the life-cycle permanent income hypothesis. The coefficients of the self-employment variable are negative in the consumption equation(s) and positive in the savings rate

[11] The equations were also estimated by robust and quantile estimators but that did not change the overall pattern of results in terms of the shadow economy variable. Also including the Non-European countries (Colombia & Mexico) into the sample does not mark any noticeable difference in the results.

equation(s) even if we exclude the shadow economy variable from the estimating equations.[12] Clearly, this result is at variance with the idea utilised by e.g. Pissarides and Weber (1989), assuming that the self-employed underreport their income. If the size of the self-employed population increases, we would expect it to show in positive consumption (and negative savings rate) effects, but that does not seem to be the case. As regards the other control variables, in most cases they follow the intuitive lines, although at times the coefficients are quite sensitive to the fixed effects specification. One reason for that is that these variables (income level, self-employment, share of agriculture) are highly autocorrelated and also highly correlated with each other, which makes individual coefficients less reliable.

4. Concluding remarks

Our analyses show that the most commonly used measure of the shadow economy is inconsistent with the idea that this kind of economy biases household income more than household consumption. Although it is true that in several countries the shadow economy measures correspond to the differences between income and consumption, for most countries the differences between disposable income and consumption expenditure do not correspond to the shadow economy data in the panel setting. This obviously does not mean that the National Accounts consumption and income data are equally prone to the shadow economy, nor that they are free from any impact of the shadow economy.

There are several caveats relating to this outcome. First of all, we focused only on the household sector. Even more importantly, there are differences in the measures of the shadow economy. The data that we used deviate quite significantly from various national measures of the shadow economy. The values used are generally much higher than the national measures, but since they do not follow a uniform conceptual and measurement pattern, it is difficult to say anything about the country and/or period by period differences. Hopefully, more alternative datasets will become available for both analytical and descriptive purposes.

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[12] The share of self-employment and shadow variables are positively correlated with $r = 0.30$, but multicollinearity is not the reason for the 'wrong' coefficient signs.

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Appendix

Table A1. The definitions and data sources

Variable name	Definition	Source
agr	share of agriculture	World Bank
CPI	consumer price index	AMECO & World Bank data banks
CV	private consumption expenditure	AMECO & IMF data banks
$debt$	household indebtedness	Eurostat
emp	share of self-employment	World Bank
$GDPus$	GDP in USD	World Bank
h	share of shadow economy	Medina and Schneider (2019)
pop	population	IMF
rem	iflow of remittances in USD (scaled by $GDPus$)	World Bank
rex	outflow of remittances in USD (scaled by $GDPus$)	World Bank
rr	real interest rate	World Bank
sb	gross household savings rate	AMECO data bank
sn	net household savings rate	AMECO data bank
YD	household disposable income (net & gross)	AMECO data bank

Source: authors' work.

Table A2. Descriptive statistics

Variable	Mean	Median	Maximum	Minimum	Std. Dev.
agr, %	9.01	6.25	45.21	1.00	7.55
$\pi = \Delta log(CPI)$	0.0511	0.0245	1.3705	−0.1864	0.1083
$\Delta c = \Delta log(CV/CPI)$	0.0226	0.0236	0.9263	−0.7507	0.0716
$debt$, %	86.32	77.35	269.77	0.27	59.98
emp, %	17.80	15.08	53.61	0.86	10.24
$GDPus$, USD	497558	197483	3893959	3788	753910
h, %	21.09	20.20	55.70	5.10	10.21
$\Delta pop = \Delta log(pop)$	0.0029	0.0029	0.0304	−0.0565	0.0089
rem, USD	0.0237	0.0061	0.2795	0.0000	0.0422
rex, USD	0.0328	0.0022	1.0948	0.0000	0.1095
rr, %	2.12	2.72	139.81	−91.72	11.44
sb; %	9.87	10.49	25.74	−19.80	6.76
sn, %	4.34	5.28	28.66	−39.35	7.94
$\Delta y = \Delta log(YD/CPI)$	0.0132	0.0174	0.2504	−0.8462	0.0771

Source: authors' work.

The household income, savings and consumption data of the following countries: Belarus, Bulgaria, Ukraine and Russia came from the respective national statistical offices. The data for Malta came from Grech (2013, pp. 42–48). In the above cases, the time-series cover much shorter time periods than the other data that, with a few exceptions, cover 1991–2017.

The set of countries consists of the following:
Austria, Belgium, Bulgaria, Croatia, Cyprus, the Czech Republic, Denmark, Estonia, Finland, France, Germany, Greece, Hungary, Iceland, Ireland, Italy, Latvia, Lithuania, Luxembourg, Malta, the Netherlands, Norway, Poland, Portugal, Romania, Slovakia, Slovenia, Spain, Sweden, Switzerland, the United Kingdom, Belarus, Russia, Ukraine; Mexico, Colombia.

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Estimation of Value-at-Risk using Weibull distribution – portfolio analysis on the precious metals market

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Abstract. In this paper, we present a modification of the Weibull distribution for the Value-at-Risk (VaR) estimation of investment portfolios on the precious metals market. The reason for using the Weibull distribution is the similarity of its shape to that of empirical distributions of metals returns. These distributions are unimodal, leptokurtic and have heavy tails. A portfolio analysis is carried out based on daily log-returns of four precious metals quoted on the London Metal Exchange: gold, silver, platinum and palladium. The estimates of VaR calculated using GARCH-type models with non-classical error distributions are compared with the empirical estimates. The preliminary analysis proves that using conditional models based on the modified Weibull distribution to forecast values of VaR is fully justified.

Keywords: risk analysis, Value-at-Risk, metals market, GARCH-type models, two-sided Weibull distribution

JEL: C32, C58, G11, G17

1. Introduction

The last decade saw a growing interest in other forms of investment than those offered by the capital market, which is mainly the effect of the uncertainty and unpredictability of the global economy trends. The crisis of 2008–2009 caused some investors to transfer their capital to other, alternative markets, in order to minimise the risk involved in their investment activity. One of these alternative markets is the metals market. The level of the volatility of metals returns depends on the moods observed on the market and is directly related to the uncertainty of the trends of many economic indicators and the occurrence of unpredictable random events that may affect these trends. Moreover, uncertainty produces risk that the future return will be below the expected level. Risk is therefore a random variable and its level is determined by measures defined for this variable.

2. Value-at-Risk

In the literature there are numerous studies on risk measurement, many of which concern Value-at-Risk (VaR). VaR has been proposed as a measure of risk by the RiskMetrics Group (the leading provider of risk management and corporate governance products and services to financial market participants). Daníelsson et al. (2013)

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examined certain properties of VaR, which showed that the VaR risk measure is subadditive in the respective tail region of the return distribution. The authors also observed that the VaR estimates calculated using the historical simulation method can lead to a violation of the subadditivity assumption. As a result, they suggested estimating VaR by means of the semi-parametric extreme value theory (EVT). Alexander and Sarabia (2012) proposed to estimate risk related to VaR and to adjust its estimates to the estimation error and model specification. Chinhamu et al. (2015) predicted the values of VaR using EVT and the generalised Pareto distribution (GPD). Other researchers analysed the quality of VaR forecasts using GARCH-type models, e.g. Chkili et al. (2014), who applied non-linear FIAPARCH models. Yu et al. (2018) measured values of VaR using GARCH-type models and EVT jointly with copula models. The results of the backtesting showed that the GARCH-EVT and copula models were able to increase the accuracy of VaR estimations. In contrast, Cheung and Yuen (2020) introduced an uncertainty model for the distribution of returns and examined the impact of this uncertainty on VaR through the worst-case scenario approach. The researchers proved that the selection of a loss model is essential when applying an uncertainty model.

Value-at-Risk is defined as a statistical measure which indicates (in an explicit manner) the amount of a potential loss of market value of a financial asset, for which the probability of reaching or exceeding this value within a specified time horizon is equal to the tolerance level determined by the decision-maker (Doman & Doman, 2009; Dowd, 1999; Trzpiot, 2004). Another definition of VaR sees it as a measure of the maximum loss that an individual can incur within a certain time horizon for an investment realised under normal market conditions, within a predefined tolerance level (Krawczyk, 2017). Assuming random variable X, the mathematical definition of VaR is as follows:

$$VaR_\alpha(X) = inf\{x|F_X(x) \geq \alpha\} = F_X^{-1}(\alpha), \tag{1}$$

where $F_X^{-1}(\alpha)$ is the quantile function of random variable X, and α is the level of the quantile of the probability distribution of this random variable. In particular, random variable X may represent return r_t of any financial asset at time t.

The advantage of defining VaR through the quantile function is the possibility to apply any probability distribution of a random variable to estimate its value. Thus, the selection of a suitable probability distribution is crucial. Empirical studies on financial data show that time series are characterised by a high level of volatility, clustering of variance, significant skewness and leptokurtosis, and the presence of outliers. These features explicitly exclude the possibility of estimating VaR through symmetrical distributions, such as normal or Student's t-distribution. Therefore, in empirical analyses, it is necessary to use probability distributions which take into consideration the above-mentioned characteristics.

3. Two-sided Weibull distribution

In this study, we propose the Weibull distribution as the theoretical tool for estimating VaR. This distribution belongs to the family of extreme distributions; therefore, it considers the presence of outliers in time series, which results in a high level of asymmetry, kurtosis and heavy tails. Technically, random variable X is described by the Weibull distribution if its density function takes the following form:

$$f(x;k,\lambda) = \begin{cases} \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} e^{\left(\frac{x}{\lambda}\right)^k} & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0 \end{cases}, \tag{2}$$

where $k > 0$ is the shape parameter and $\lambda > 0$ is the scale parameter. The density function given by formula (2) can also be defined as

$$f(x;k,b) = \begin{cases} bkx^{k-1}e^{-bx^k} & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0 \end{cases}, \tag{3}$$

where $b = \lambda^{-k}$ is the scale parameter.

As mentioned above, the Weibull distribution is applied in EVT and therefore can be used to describe rare events which significantly affect the estimates of the tail risk measure for a relatively low level of the quantile. Formulas (2)–(3) demonstrate that the density function of the Weibull distribution is equal to zero for negative values of random variable X. Chen and Gerlach (2013) proposed a certain generalisation of the classical (one-sided) Weibull distribution over the entire set of real numbers by introducing a standardised two-sided Weibull distribution, for which the density function has the form of

$$f_{\mathrm{DW}}(x;k_1,\lambda_1,k_2,\lambda_2) = \begin{cases} b_p\left(\frac{-b_p x}{\lambda_1}\right)^{k_1-1} \exp\left[-\left(\frac{-b_p x}{\lambda_1}\right)^{k_1}\right] & \text{if} \quad x < 0 \\ b_p\left(\frac{b_p x}{\lambda_2}\right)^{k_2-1} \exp\left[-\left(\frac{b_p x}{\lambda_2}\right)^{k_2}\right] & \text{if} \quad x \geq 0 \end{cases}, \tag{4}$$

where $k_1, k_2 > 0$ are shape parameters and $\lambda_1, \lambda_2 > 0$ are scale parameters. In addition,

$$b_p^2 = \frac{\lambda_1^3}{k_1}\Gamma\left(1+\frac{2}{k_1}\right) + \frac{\lambda_2^3}{k_2}\Gamma\left(1+\frac{2}{k_2}\right) - \left[-\frac{\lambda_1^2}{k_1}\Gamma\left(1+\frac{1}{k_1}\right) + \frac{\lambda_2^2}{k_2}\Gamma\left(1+\frac{1}{k_2}\right)\right]^2 \tag{5}$$

and $\frac{\lambda_1}{k_1} + \frac{\lambda_2}{k_2} = 1$.

The estimates of VaR using two-sided Weibull distribution can be obtained by using the quantile function:

$$VaR_\alpha = F^{-1}(\alpha; k_1, \lambda_1, k_2, \lambda_2) = \begin{cases} -\frac{\lambda_1}{b_p}\left[-\ln\left(\frac{k_1}{\lambda_1}\alpha\right)\right]^{\frac{1}{k_1}} & \text{if} \quad 0 \leq \alpha < \frac{\lambda_1}{k_1} \\ \frac{\lambda_2}{b_p}\left[-\ln\left(\frac{k_2}{\lambda_2}(1-\alpha)\right)\right]^{\frac{1}{k_2}} & \text{if} \quad \frac{\lambda_1}{k_1} \leq \alpha < 1 \end{cases}. \tag{6}$$

Considering quantile function for returns r_t, a one-day-ahead VaR forecast of α-quantile is defined as

$$\alpha = P(r_{t+1} < VaR_\alpha | \mathrm{I}_t), \tag{7}$$

where r_{t+1} is the return at time $t+1$, α is the level of the quantile, and I_t is the information set at time t. Consequently, resulting from the above, VaR is defined as the α-quantile of a conditional distribution of r_t.

4. Empirical study

Metals are commodities used in many areas of human activity. These include heavy industry (military, construction and infrastructure), aerospace (spacecraft, orbital probes, telescopes) and the automotive industry (production of cars and car components). Metals are used in the production of household appliances, they are also used as alloys in various steel compounds, mainly to improve their quality and expand their physical properties. Metals are not only related to industry, but they are also used in the jewellery trade (mainly precious metals), medicine (including aesthetic), biotechnology and in gastronomy (gold and silver). From an investment point of view, metals, being commodities quoted on stock exchanges, can be the subject of financial investments (direct and indirect). The above refers primarily to precious metals, which are an alternative form of investing if compared to the classical capital market assets, such as stocks or bonds.

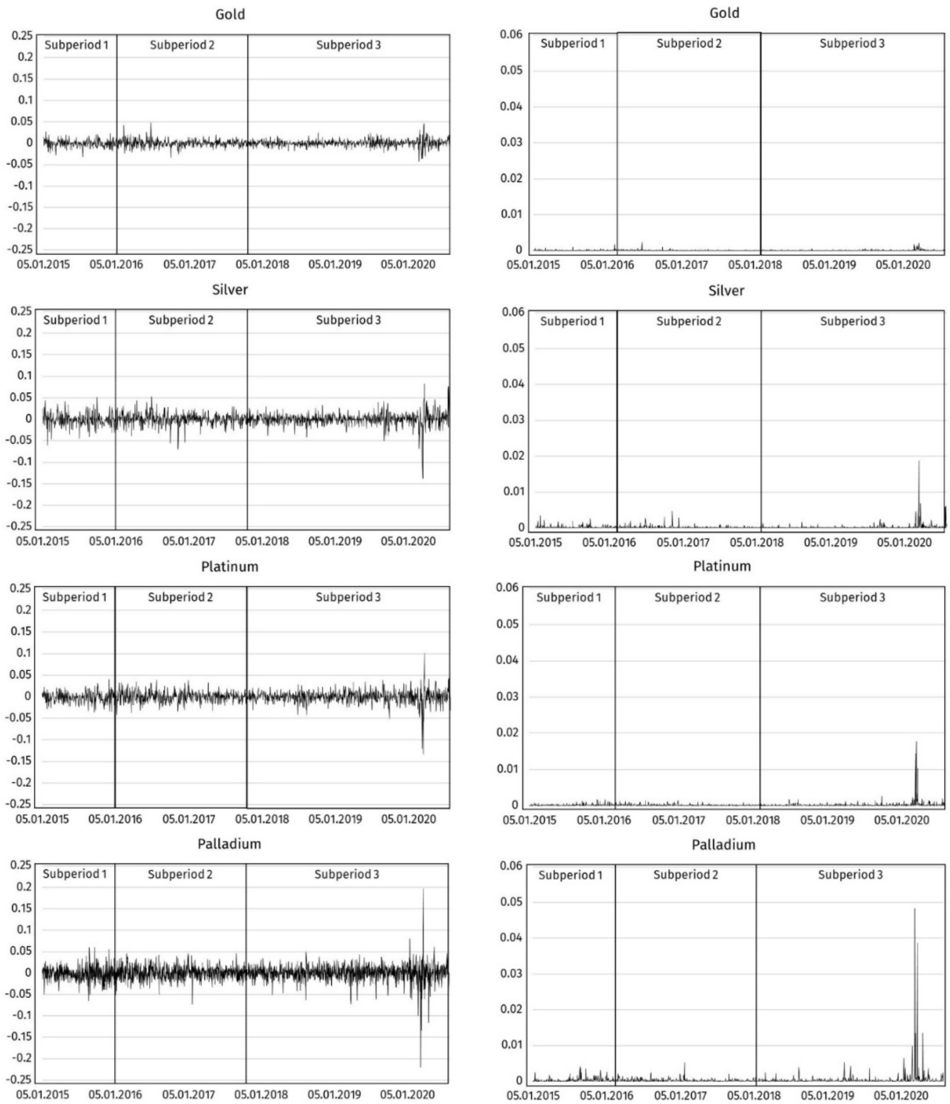
The metals market is not a popular area of interest among researchers, although the number of papers on risk analysis in this area has clearly increased in the recent years. However, research is mainly concerned with gold. Zijing and Zhang (2016) analysed the volatility and risk of precious metals returns using GARCH-type models with a random error described by the GED distribution, while Włodarczyk (2017) analysed the impact of asymmetry and long memory effects on forecasting conditional volatility and the risk of gold and silver using linear and non-linear GARCH models. Chen and Qu (2019) analysed the risk and volatility of precious metals returns using copula and dynamic conditional correlation (DCC) models. In turn, Krężołek (2020) has conducted extensive research on risk modelling of the base and precious metals markets. The author showed in his research, among other things, that fat-tailed distributions (including alpha-stable distributions) and ARMA-GARCH-type models should be used for risk modelling. Other methods were proposed by Wang et al. (2019), who predicted the volatility and risk of copper prices by comparing complex hybrid networks with traditional artificial neural network techniques. The results demonstrated that the proposed hybrid models were able to achieve a favourable predictive effect both in forecasting the levels of risk and volatility in copper prices.

In this study we use daily log-returns of four precious metals: gold, silver, platinum and palladium for the construction of investment portfolios. The data come from the London Metal Exchange (LME) from the period of January 2015–July 2020, which has further been divided into three sub-periods:

- sub-period 1 (2015): portfolio construction;
- sub-period 2 (2016–2017): model estimation;
- sub-period 3 (2018–2020): forecasting of VaR.

The main goal of this research is to estimate the Value-at-Risk of investment portfolios using selected models of conditional volatility (GARCH-type models) with error terms described by the following non-classical probability models: Student's t-distribution, skewed Student's t-distribution, GED, skewed GED and the two-sided Weibull distribution. Four investment portfolios have been constructed, for which the values of VaR (for the quantile of 0.01 and 0.05) have been estimated, according to the proposed theoretical model. The quality of the forecasts has been assessed using the test of exceedance proposed by Kupiec (1995) and the independence test introduced by Christoffersen (1998). Figure 1 presents returns and squares of returns of all the studied precious metals, while Table 1 presents the descriptive statistics of returns by sub-periods.

Figure 1. Log-returns (left) and squares of log-returns (right) of selected precious metals between January 2015 and July 2020



Source: author's work based on data from LME.

The first and second sub-period saw a comparatively stable level of variance, while in the third sub-period a relatively significant clustering of volatility was observed (early 2020), which resulted from the socio-economic condition in the worldwide economy caused by the COVID-19 pandemic. Moreover, the data show that gold returns, compared to other metals, did not react strongly to the information from the market during the pandemic period. This results from the fact that gold is perceived as a 'safe haven' in times of increasing uncertainty in the global economy (Salisu et al., 2021). However, some studies indicate that during the pandemic, for some assets, gold lost its 'safe haven' property (Cheema et al., 2020).

Table 1. Descriptive statistics of log-returns for three sub-periods

Statistics	Gold	Silver	Platinum	Palladium
Sub-period 1				
Mean	−0.00044	−0.00051	−0.00115	−0.00134
Standard deviation	0.00858	0.01488	0.01228	0.01862
Coefficient of variation in %	−1966.40	−2922.26	−1066.04	−1392.01
Skewness	0.00002	−0.06384	−0.07000	0.01858
Kurtosis	1.16474	1.94523	0.41096	1.11559
Minimum	−0.03280	−0.05967	−0.03641	−0.06474
Maximum	0.02712	0.05112	0.04046	0.05992
Sub-period 2				
Mean	0.00040	0.00039	0.00008	0.00124
Standard deviation	0.00819	0.01309	0.01200	0.01573
Coefficient of variation in %	2053.50	3324.77	14880.94	1271.04
Skewness	0.46418	−0.32880	0.08377	−0.23870
Kurtosis	3.86023	2.96838	0.74131	1.02741
Minimum	−0.03300	−0.06882	−0.04139	−0.07233
Maximum	0.04867	0.05258	0.03814	0.04602
Sub-period 3				
Mean	0.00061	0.00049	−0.00004	0.00101
Standard deviation	0.00821	0.01548	0.01568	0.02275
Coefficient of variation in %	1354.91	3157.36	−42793.61	2242.77
Skewness	−0.06169	−0.76277	−1.29900	−0.99925
Kurtosis	4.68469	14.09777	15.30709	24.14889
Minimum	−0.04196	−0.13719	−0.13300	−0.21994
Maximum	0.04605	0.08243	0.10163	0.19665

Source: author's calculations based on data from LME.

In the first sub-period, compared to the others, the returns of all precious metals had a negative average value. Regardless of the sub-period, a high level of volatility is observed. Additionally, the empirical distributions of returns are skewed and leptokurtic in all the sub-periods (especially in the third one). Based on the data from the first sub-period, investment portfolios of three components have been constructed in such a way that each portfolio contains a different combination of components:

- P_1 – gold, silver, platinum;
- P_2 – gold, silver, palladium;

- P_3 – gold, platinum, palladium;
- P_4 – silver, platinum, palladium.

Optimal portfolios have been determined with the assumption that there is no possibility of short selling, whereas the optimisation criterion involves the minimisation of the portfolio's risk (measured by variance). Weights of metals in optimal portfolios are presented in Table 2, whereas the expected return and risk for equally weighted and optimal portfolios are presented in Table 3 and Figure 2.

Table 2. Weights of components in optimal portfolios

Metal	$P_{1opt.}$	$P_{2opt.}$	$P_{3opt.}$	$P_{4opt.}$
	in %			
Gold	100.00	96.72	96.72	.
Silver	0.00	0.00	.	20.79
Platinum	0.00	.	0.00	73.09
Palladium	.	3.28	3.28	6.13

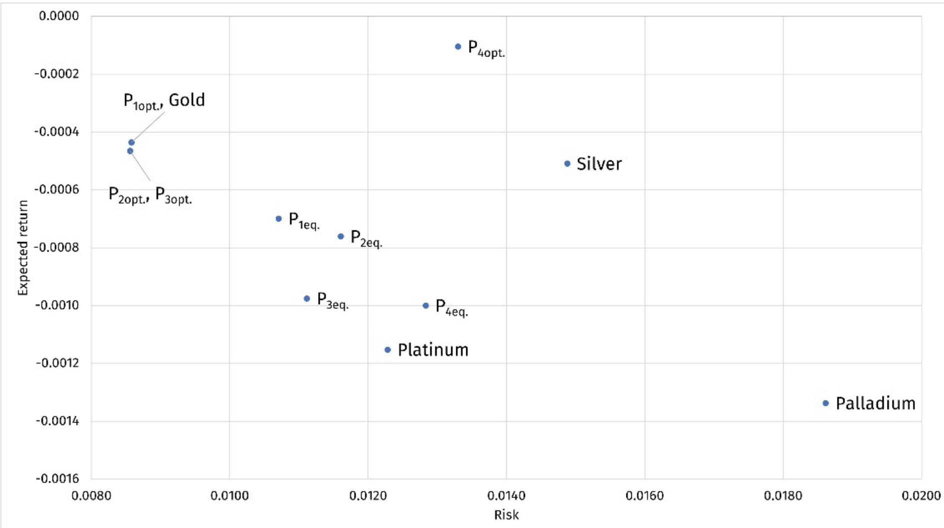
Source: author's calculations based on data from LME.

Table 3. The risk and expected return for equally weighted and optimal portfolios in sub-period 1

Portfolio	Risk	Expected return
Equally weighted		
$P_{1eq.}$	0.01071	–0.00070
$P_{2eq.}$	0.01161	–0.00076
$P_{3eq.}$	0.01112	–0.00098
$P_{4eq.}$	0.01283	–0.00100
Optimal		
$P_{1opt.}$	0.00858	–0.00044
$P_{2opt.}$	0.00856	–0.00047
$P_{3opt.}$	0.00856	–0.00047
$P_{4opt.}$	0.00010	–0.00010
Individual assets		
Gold	0.00858	–0.00044
Silver	0.01488	–0.00051
Platinum	0.01228	–0.00115
Palladium	0.01862	–0.00134

Source: author's calculations based on data from LME.

Figure 2. The risk and expected return for equally weighted and optimal portfolios, and for individual assets



Source: author's work based on data from LME.

As a result of optimisation, the level of risk decreased for all portfolios and, in addition, the expected loss was reduced. Optimal portfolios $P_{2opt.}$ and $P_{3opt.}$ have the same characteristics because the optimisation resulted in the same components for these two portfolios (in the further part of the analysis, these two portfolios are denoted as one, namely $P_{2.3opt.}$). Moreover, individual investments show a higher level of risk than optimal portfolios. Gold remains the only exception, for which both a low level of risk and a relatively low level of the expected loss are observed.

In the next step of the analysis, involving data from the second sub-period, the parameters of conditional volatility models for optimal portfolio returns have been estimated at GARCH(1,1) and APARCH(1,1) for different error distributions. The conditional variance equations for the GARCH (Bollerslev, 1986) and APARCH models (Ding et al., 1993) are denoted by the following formulas:

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i a_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2, \tag{8}$$

$$\sigma_t^\delta = \alpha_0 + \sum_{i=1}^{q} \alpha_i (|a_{t-i}| - \gamma_i a_{t-i})^\delta + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^\delta, \tag{9}$$

where $\alpha_0 \geq 0$, $\alpha_i \geq 0$ for $i > 0$, $\beta_j \geq 0$, $\sum_{i=1}^{max(p,q)} (\alpha_i + \beta_i) < 1$, $\varepsilon_t \sim N(0,1)$ and ε_t is iid. Based on the characteristics of the time series of metals returns

(Krężołek, 2020), the following error distributions for conditional models are proposed:

- Student's t-distribution:

$$f_{\mathrm{st.}}(\varepsilon_t, \sigma_t^2; \theta) = \frac{\Gamma\left(\frac{v+1}{2}\right)}{\sigma_t \Gamma\left(\frac{v}{2}\right)\sqrt{\pi(v-2)}}\left(1 + \frac{\varepsilon_t^2}{(v-2)\sigma_t^2}\right)^{\frac{v+1}{2}}, \tag{10}$$

where v is the number of degrees of freedom, and $\Gamma(k) = \int_0^{+\infty} x^{k-1}e^{-1}\,dx$ is a gamma function with parameter k;

- Skewed Student's t-distribution:

$$f_{\mathrm{sst.}}(x, v) = \frac{2}{\zeta + \frac{1}{\zeta}}\{g(\zeta(ax+b); v)I_{x<-\frac{b}{a}} + g\left(\frac{ax+b}{\zeta}; v\right)I_{x\geq -\frac{b}{a}}, \tag{11}$$

where $a = \frac{\Gamma\left(\frac{v-1}{2}\right)\sqrt{v-2}}{\sqrt{\pi}\Gamma\left(\frac{v}{2}\right)}\left(\zeta - \frac{1}{\zeta}\right)$, $b^2 = \left(\zeta^2 + \frac{1}{\zeta^2} - 1\right) - a^2$; ζ is the skewness parameter, and $g(\cdot)$ is the density function of a standard Student's t-distribution with v degrees of freedom;

- GED distribution:

$$f_{\mathrm{GED}}(\varepsilon_t, \sigma_t^2; \theta) = 2^{-\frac{v+1}{v}} \frac{v}{\sigma_t\sqrt{\frac{\Gamma(v^{-1})}{\Gamma(3v^{-1})}2^{-\frac{2}{v}}\Gamma(v^{-1})}} exp\left\{-\frac{1}{2}\left|\frac{\varepsilon_t}{\sigma_t\sqrt{\frac{\Gamma(v^{-1})}{\Gamma(3v^{-1})}2^{-\frac{2}{v}}}}\right|^v\right\}, \tag{12}$$

where v is the number of degrees of freedom, and $\Gamma(k) = \int_0^{+\infty} x^{k-1}e^{-1}\,dx$ is a gamma function with parameter k;

- Skewed GED distribution:

$$f_{\mathrm{sGED}}(x) = \frac{k^{1-\frac{1}{k}}}{2\sigma}\Gamma\left(\frac{1}{k}\right)^{-1}\exp\left(-\frac{1}{k}\frac{|u|^k}{(1+\mathrm{sgn}(u)\zeta)^k\sigma^k}\right), \tag{13}$$

where $u = x - m$ (m – the mode of random variable X), σ is the scale parameter, ζ is the skewness parameter, k is the kurtosis parameter, $\mathrm{sgn}(\cdot)$ is the sign function, v is the number of degrees of freedom, and $\Gamma(k) = \int_0^{+\infty} x^{k-1}e^{-1}\,dx$ is a gamma function with parameter k.

In addition, the standard two-sided Weibull distribution has also been applied, with the density function given by formula (4). The final model of the conditional volatility for a given optimal portfolio has been selected using the Akaike Information Criterion (AIC). The values of the AIC criterion are presented in Table 4.

Table 4. AIC information criterion for GARCH and APARCH models for optimal portfolios

Model	$P_{1opt.}$	$P_{2.3opt.}$	$P_{4opt.}$
$GARCH_{st.}(1,1)$	−5452.99[a]	−5457.55	−4823.82
$GARCH_{sst.}(1,1)$	−5451.01	−5457.98	−4823.37
$GARCH_{GED}(1,1)$	−5449.08	−5455.38	−4824.32
$GARCH_{sGED}(1,1)$	−5447.11	−5453.39	−4825.39
$GARCH_{dW}(1,1)$	−5447.98	−5458.43[a]	−4829.65[a]
$APARCH_{st.}(1,1)$	−5455.49[b]	−5460.55[b]	−4820.54
$APARCH_{sst.}(1,1)$	−5453.62	−5458.63	−4820.06
$APARCH_{GED}(1,1)$	−5450.34	−5455.45	−4820.72
$APARCH_{sGED}(1,1)$	−5448.39	−5453.51	−4821.83
$APARCH_{dW}(1,1)$	−5453.86	−5419.51	−4829.76[b]

a The lowest value of AIC for GARCH models. b The lowest value of AIC for APARCH models.
Source: author's calculations based on data from LME.

The GARCH and APARCH models with error terms described by Student's t-distribution were selected for the first portfolio $P_{1opt.}$. For portfolio $P_{2.3opt.}$, the most convenient GARCH model is the one with an error term described by the two-sided Weibull distribution and the APARCH model with an error term described by Student's t-distribution. The GARCH and APARCH models with error terms described by two-sided Weibull distribution were selected for the last portfolio $P_{4opt.}$.

In the last phase of the study, one-day-ahead VaR forecasts are calculated for the data from the third sub-period. For this purpose, models of conditional volatility selected on the basis of the AIC criterion have been used. The verification of the number of exceedances has been carried out on the average VaR forecasts from the third sub-period for all optimal portfolios using the Kupiec (LR_{POF}) and Christoffersen (LR_{IND}) tests. All results are presented in Table 5 ($VaR_{0.01}$) and 6 ($VaR_{0.05}$).

Table 5. Average one-day-ahead $VaR_{0.01}$ forecasts within the third sub-period (Kupiec test and Independence test)

Volatility model	$VaR_{0.01}$	% of failure	Kupiec test		Independence test	
			LR_{POF}	p-value	LR_{IND}	p-value
			$P_{1opt.}$			
Empirical	−0.02968	0.00978	0.00202	0.96419	0.82214	0.36346
$GARCH_{st.}(1,1)$	−0.03196	0.00733	0.32334	0.56961	1.15428	0.28265
$APARCH_{st.}(1,1)$	−0.03041	0.00978	0.00202	0.96419	1.73445	0.18784
			$P_{2.3opt.}$			
Empirical	−0.02817	0.00978	0.00202	0.96419	0.82214	0.36346
$GARCH_{dW}(1,1)$	−0.02834	0.00978	0.00202	0.96419	0.82214	0.36346
$APARCH_{st.}(1,1)$	−0.02761	0.01222	0.19098	0.66211	2.51775	0.11257
			$P_{4opt.}$			
Empirical	−0.04378	0.00978	0.00202	0.96419	0.82214	0.36346
$GARCH_{dW}(1,1)$	−0.04715	0.00978	0.00202	0.96419	1.15428	0.28265
$APARCH_{dW}(1,1)$	−0.04337	0.01222	0.19098	0.66211	2.51775	0.11257

Source: author's calculations based on data from LME.

Table 6. Average one-day-ahead $VaR_{0.05}$ forecasts within the third sub-period (Kupiec test and Independence test)

Volatility model	$VaR_{0.05}$	% of failure	Kupiec test		Independence test	
			LR_{POF}	p-value	LR_{IND}	p-value
			$P_{1opt.}$			
Empirical	−0.01474	0.04890	0.01050	0.91840	0.71170	0.39888
$GARCH_{st.}(1,1)$	−0.01538	0.04156	0.64838	0.42069	1.78352	0.18172
$APARCH_{st.}(1,1)$	−0.01489	0.04645	0.11074	0.73931	1.17532	0.27831
			$P_{2.3opt.}$			
Empirical	−0.01335	0.04890	0.01050	0.91840	0.71170	0.39888
$GARCH_{dW}(1,1)$	−0.01494	0.04156	0.64838	0.42069	1.78352	0.18172
$APARCH_{st.}(1,1)$	−0.01403	0.04890	0.01050	0.91840	0.92672	0.33572
			$P_{4opt.}$			
Empirical	−0.02177	0.04890	0.01050	0.91840	0.71170	0.39888
$GARCH_{dW}(1,1)$	−0.02245	0.04645	0.11074	0.73931	0.01589	0.89969
$APARCH_{dW}(1,1)$	−0.02293	0.04645	0.11074	0.73931	0.01589	0.89969

Source: author's calculations based on data from LME.

The empirical forecasts of VaR for optimal portfolios differ depending on the model and the quantile level. VaR forecasts estimated using GARCH models, regardless of the assumed probability distribution for the error, were overestimated, while forecasts estimated using APARCH models were usually underestimated. Using the convergence criterion as the minimum value of the root mean square error (RMSE), the APARCH models allowed the estimation of the forecasts of VaR at a level relatively close to the empirical estimates. Referring to the results obtained in the context of the probability distribution for the error term, the models estimated by using two-sided Weibull distribution provided correct and accurate predictions of VaR. This was also confirmed by the results of the Kupiec and Christoffersen tests.

5. Conclusions

In this study we proposed the application of the two-sided Weibull distribution to forecast the values of VaR for investment portfolios on the precious metals market. A selection of conditional volatility models was used. The choice to apply the Weibull distribution resulted from the observed properties of precious metals' returns, including high-level volatility, clustering of variance, asymmetry and kurtosis, as well as the existence of outliers, which significantly affect the values of probability measured in the tail of the distribution. GARCH and APARCH models with non-classical error distributions were selected to describe the conditional volatility. The analysis was carried out for daily log-returns of four precious metals quoted on the LME between January 2015 and July 2020. This period was divided into three sub-periods, namely the construction of portfolios, model estimation and the forecasting of VaR. VaR was estimated at the quantile level of 0.01 and 0.05 for portfolio returns.

The results of the analysis show that the optimisation of portfolios on the precious metals market led to a simultaneous reduction in the level of risk and in the value of expected loss. The application of the AIC information criterion allowed the selection of conditional volatility models for each of the portfolios; these models had error terms described by Student's t-distribution and two-sided Weibull distributions. In the last phase of the research, one-day-ahead VaR forecasts were calculated on the basis of selected models. It was observed that, regardless of the error distribution, GARCH models overestimated and APARCH models underestimated the empirical values of VaR. The study also proved that the VaR estimates were accurate due to the use of models with an error term described by the two-sided Weibull distribution, which was confirmed by the Kupiec and Christoffersen tests. In conclusion, the two-sided Weibull distribution is an appropriate theoretical tool to determine forecasts of Value-at-Risk for investment portfolios on the precious metals market.

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