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Ascending Probabilistic Max-min Extended Choice Correspondence

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Abstract. In this paper, we provide an axiomatic characterization of the ascending probabilistic max-min extended choice correspondence for a decision-maker who has state-dependent preferences (represented by a linear order) over a set of alternatives and a (subjective) probability vector over states of nature, where both the preferences and probability vectors are variable. Further on the domain of all extended preference profiles for which the Ascending Probabilistic Max-Min Extended Choice Correspondence is resolute, the same choice correspondence is completely characterized by just two of the three axioms that are required for the axiomatic characterization on the more general domain. A significant feature of our solution concept and the related axiomatic analysis is that we use no more information than the probability with which each alternative realizes each rank.

Keywords: decision-making under risk, state-dependent preferences, extended choice correspondences, ascending probabilistic max-min
JEL: C02, D81, D91

1. Introduction

The framework of decision-making under uncertainty, introduced in Lahiri (2020/2021, 2022), is that of a decision-maker who is faced with making a choice under probabilistic uncertainty (risk) regarding the future state of nature, which is realized after the decision has been made. The decision-maker is provided with (or aware of) an extended preference profile, which is a pair whose first component is a profile of state-dependent rankings over a non-empty finite set of alternatives (the consequences) and whose second component is a probability distribution over a non-empty finite set of states of nature. A decision support system (DSS) or decision aid is required to choose a non-empty "desirable" set of alternatives from which the final choice has to be made. The decision aid or DSS has no bias in favor of any one or more alternatives that it suggests. Such a decision support system is called an extended choice correspondence, i.e. a rule which associates each extended preference profile from a given set of extended preference profiles with a non-empty finite set of desirable alternatives. For related literature, one might wish to consult Lahiri (2022).

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Here we begin by setting up the model for extended choice correspondences. In this framework, we define and provide an axiomatic characterization of the Ascending Probabilistic Max-min Extended Choice Correspondence, which is a refined version of the Probabilistic Max-min Choice Correspondence defined and axiomatically characterized in Lahiri (2022). The Probabilistic Max-min Extended Choice Correspondence is based on the Max-min Choice Correspondence defined in Campbell, Kelly and Qi (2018). This choice correspondence selects, for each preference profile, those alternatives which have the best "worst rank". In our framework, for an extended preference profile – a pair comprising of a strict preference profile and a probability vector (for the states of nature) – a "max-min alternative" is an alternative whose worst rank among states of nature that occur with positive probability is the best. The worst rank of a max-min alternative is said to be the "max-min rank". The probabilistic max-min extended choice correspondence selects, for each extended preference profile, those max-min alternatives which have the least positive probability of attaining the "max-min rank". We ignore those states of nature which occur with zero probability, since if an alternative attains its worst rank with zero probability, it is improbable (though not impossible) that it will attain such a rank. Further, if a max-min alternative attains the max-min rank with the lowest probability, then it attains a superior rank with the highest probability among all the max-min alternatives. The Ascending Probabilistic Max-min Extended Choice Correspondence chooses those alternatives from among the Probabilistic Max-min winners that occur with the greatest (cumulative) probability of a better rank, as we keep improving the rank, one rank each time, and stop as soon as we arrive at a unique solution, or the moment we reach the first rank – whichever happens sooner. It is very unlikely that a risk-averse individual to whom the probabilistic max-min extended choice correspondence is recommended would wish for anything better. Hence, the solution studied here unconditionally supersedes the solution presented in Lahiri (2022).

The axioms we use to characterize the Ascending Probabilistic Max-min Extended Choice Correspondence are Independence of Irrelevant States, Probabilistic Neutrality and No-Terminal Stochastic Domination. Further on the domain of all extended preference profiles for which the Ascending Probabilistic Max-Min Extended Choice Correspondence is resolute, the same choice correspondence is completely characterized by Independence of Irrelevant States and No-Terminal Stochastic Domination. Probabilistic Neutrality is no longer required for the axiomatic characterization on such a domain.

The Independence of Irrelevant States says that states of nature that occur with zero probability have no influence or effect on the choice procedure. Probabilistic Neutrality says that if two alternatives have identical probabilities of realizing each and every rank, then either both are chosen or neither is chosen. An alternative is said to terminally stochastically dominate another alternative if there is a rank such that the probability of the first alternative getting that rank or better is higher than the probability of the second alternative getting the same rank or better, and for all worse ranks, the probability of the first alternative getting that rank or better is no less than the probability of the second alternative getting the same rank or better. In other words, towards the end the first alternative has a better chance of having a preferred position than the second alternative. No-Terminal Stochastic Domination says that a terminally stochastically dominated alternative is not chosen. The interesting characteristic of our result is that we are able to obtain it without any axiom appealing to worst ranks, although the worst rank is one of the most important features – in fact the starting point – in the definition of our solution concept. A significant feature of our solution concept and the related axiomatic analysis is that we use no more information than the probability with which each alternative realizes each rank.

The domain of the ascending probabilistic max-min extended choice correspondence whose axiomatization we provide is the set of all extended preference profiles such that, for any non-empty subset of probability vectors, all strict preference profiles can be associated with any probability vector in the subset. However, the axiomatic characterization we provide continues to remain valid on the strict sub-domain where the extended preference profiles are such that those states of nature that occur with positive probability have an equal probability of occurrence. Such a domain is called a domain with equiprobable support. As regards the domain with equiprobable support, our solution concept is a strict refinement of the one discussed in Campbell, Kelly and Qi (2018), with a different interpretation.

2. The framework of the analysis

The following framework, which is identical to the one in Lahiri (2022), is a relatively close adaptation of the ones from Denicolò (1985), Section 2.2 of Endriss (2011) and those discussed thoroughly in Lahiri (2020/2021). There are passages in Sections 2 and 3 of the latter paper where the wording is identical to some passages in Lahiri (2022). It is necessary to include them, since unlike the results which can be referred to, these passages are concerned with basic notations and definitions, and it would be a harassment for the readers to ask them to look for those definitions elsewhere. However, all such passages have been included between inverted commas in what follows.

"Consider a decision-maker (DM) faced with the problem of choosing one or more alternatives from a non-empty finite set of alternatives X, containing at least three elements. Let $\Psi(X)$ denote the set of all non-empty subsets of X. For a positive integer $n \geq 3$, let $N = \{1,2,\ldots,n\}$. In contrast to the convention, we will refer to an element in N as a state of nature and the set N as the set of states of nature.

A strict preference relation/strict ranking on X is a linear order (i.e. a reflexive, complete/connected/total, transitive and anti-symmetric binary relation) on X. Generally, a strict preference relation is denoted by R with P signifying its asymmetric part. If for $x, y \in$ X it is the case that $(x, y) \in$ R, then we shall denote it by xRy and say that x is at least as good as y for the strict preference relation R. Similarly, xPy interpreted as x is strictly preferred to y for the strict preference relation R.

Given a strict preference R and an alternative x, the rank of x at R denoted $\text{rk}(x, \text{R}) = |\{y \in \text{X} | y\text{R}x\}|$, i.e. 1 + cardinality of the set of alternatives strictly preferred to x for the strict preference relation R.

Let $\mathcal{L}$ denote the set of all strict preference relations on X."

A convenient way to display/represent a strict ranking R is by using an m-dimensional column vector $\begin{pmatrix} x \\ \vdots \\ z \end{pmatrix}$, such that the entry in the r[th] row corresponds to the alternative that has the r[th] rank at the strict ranking R.

"A strict preference profile denoted R_N is a function from N to $\mathcal{L}$. R_N is represented as the array $\langle \text{R}_\text{i} | \text{i} \in \text{N} \rangle$, where R_i is the strict preference relation/strict ranking in state of nature i. The set of all preference profiles is denoted $\mathcal{L}^\text{N}$.

A probability vector over N is a vector $\text{p} \in \mathbb{R}_+^\text{N}$ satisfying $\sum_{\text{i}=1}^{\text{N}} \text{p}_\text{i} = 1$, where for $\text{i} \in \text{N}$, p_i is the probability that state of nature i occurs.

The set of probability vectors over N is denoted by Δ.

Given a probability vector p, the set $\{\text{j} | \text{p}_\text{j} > 0\}$ is referred to as the support of p and denoted support(p).

Since probabilities are associated with events, for each $\text{i} \in \text{N}$, the state of nature i represents a non-empty set, and N is a finite partition of some underlying sample space.

A pair $(\text{R}_\text{N},\text{p}) \in \mathcal{L}^\text{N} \times \Delta$ is said to be an extended preference profile and $\mathcal{L}^\text{N} \times \Delta$ is the set of all extended preference profiles.

Given $(\text{R}_\text{N},\text{p}) \in \mathcal{L}^\text{N} \times \Delta$ and an alternative x (i.e. $\text{x} \in \text{X}$), the state of nature i (i.e. $\text{i} \in \text{N}$) is referred to as the worst state of nature for x at $(\text{R}_\text{N},\text{p})$ if $\text{i} \in \underset{\text{j} \in \text{support(p)}}{\text{argmax}} \text{rk}(\text{x}, \text{R}_\text{j})$.

The definition above says that a state of nature is the worst state of nature for an alternative if the state of nature occurs with "positive probability", and the alternative does not attain any worse rank with "positive probability".

Given $(\text{R}_\text{N},\text{p}) \in \mathcal{L}^\text{N} \times \Delta$ and an alternative x (i.e. $\text{x} \in \text{X}$), the set $\text{WS}(\text{x}, (\text{R}_\text{N},\text{p})) = \{\text{i} |$ i is the worst state of nature for x$\}$ is said to be the set of the worst states of nature for x at $(\text{R}_\text{N},\text{p})$, and for $\text{i} \in \text{WS}(\text{x}, (\text{R}_\text{N},\text{p}))$, $\text{rk}(\text{x},\text{R}_\text{i})$ denoted $\text{worstrk}(\text{x},(\text{R}_\text{N},\text{p}))$ is said to be the worst rank of x at $(\text{R}_\text{N},\text{p})$.

Clearly, $\text{worstrk}(\text{x},(\text{R}_\text{N},\text{p})) = \max\{\text{rk}(\text{x},\text{R}_\text{i}) | \text{i} \in \text{support(p)}\}$ for all $\text{x} \in \text{X}$.

For all $(R_N,p)\in\mathcal{L}^N\times\Delta$, let $Mm(R_N,p) = \underset{y\in X}{\text{argmin}}\, worstrk(y,(R_N,p))$.

$Mm(R_N,p)$ is said to be the set of max-min alternatives at (R_N,p). The max-min rank for (R_N,p) is equal to the unique $worstrk(x,(R_N,p))$ for any $x\in Mm(R_N,p)$.

A domain is any non-empty subset of $\mathcal{L}^N\times\Delta$. We will denote a domain by $\mathcal{R}$.

An extended choice correspondence (ECC) on (domain) $\mathcal{R}$ is a function f from $\mathcal{R}$ to $\Psi(X)$."

Useful Notations: Given $(R_N,p)\in\mathcal{L}^N\times\Delta$, $x\in X$ and $r\in\{1,\ldots,m\}$:

(a) Let $Pr(\{rk(x) = r\}|(R_N,p))$ denote the probability of x being ranked r^{th} at (R_N,p), which is equal to $\sum_{\{j\in N|rk(x,R_j)=r\}} p_j$.

(b) Let $Pr(\{rk(x) \le r\}|(R_N,p))$ denote the probability of x being ranked "r^{th} or better" at (R_N,p), which is equal to $\sum_{\{j\in N|rk(x,R_j)\le r\}} p_j$.

Given that $(R_N,p)\in\mathcal{L}^N\times\Delta$, $x\in X$ and $r\in\{2,\ldots,m\}$, let $Pr(\{rk(x) < r\}|(R_N,p))$ denote the probability of x being ranked "better than r^{th}" at (R_N,p), which is equal to $\sum_{\{j\in N|rk(x,R_j)<r\}} p_j$.

An ECC on (domain) $\mathcal{R}$ is said to be resolute if it is singleton valued for all preference profiles on $\mathcal{R}$.

3. Some axioms and a lemma which will be useful on the way

"In what follows, we will be concerned only with those domains which satisfy the following property:

Domain Property: $R=\mathcal{L}^N\times Q$, where Q is a non-empty subset of Δ."

The following is a desirable axiom that few would wish to contest.

An ECC f on $\mathcal{R}$ is said to satisfy Independence of Irrelevant States (be Independent of Irrelevant of States) (IIS) if for all (R_N,p), $(R'_N,p)\in\mathcal{R}$: $[\{j|p_j > 0\} \subset \{j|R_j = R'_j\}]$ implies $[f(R'_N,p) = f(R_N,p)]$.

In view of (IIS) and the issues we will be concerned with here – which depend only on the probability with which each strict ranking occurs – an alternative way of displaying an extended preference profile is equally convenient.

If for $(R_N,p)\in\mathcal{R}$ there exists a positive integer K such that a strict ranking $R = R_j$ for $j\in$ support (p) if and only if $R\in\{R_{(1)},\ldots,R_{(K)}\}$, then (R_N,p) can be displayed as:

$\begin{matrix} p_{(1)} & \cdots & p_{(k)} & \cdots & p_{(K)} \\ [R_{(1)} & \cdots & R_{(k)} & \cdots & R_{(K)}] \end{matrix}$, where $[R_{(1)} \;\cdots\; R_{(k)} \;\cdots\; R_{(K)}]$ is an $m \times K$ matrix such that for $k\in\{1,\ldots,K\}$, the k^{th} column is the column vector representing the strict ranking $R_{(k)}$ and the $p_{(k)}$ on top of the k^{th} column denotes the probability with which the state of nature is such that the strict ranking $R_{(k)}$ is realized, i.e., $p_{(k)} = \sum_{\{j\in support(p)|R_j=R_{(k)}\}} p_j$.

An ECC f on $\mathcal{R}$ is said to satisfy Probabilistic Neutrality if for all $(R_N,p)\in\mathcal{R}$ and $x,y\in X$: $[\sum_{\{j\in N|rk(y,R_j)=r\}} p_j = \sum_{\{j\in N|rk(x,R_j)=r\}} p_j$ for all $r\in\{1,\ldots,m\}]$ implies $[x\in f(R_N,p)$ if and only if $y\in f(R_N,p)]$.

Given that $(R_N,p)\in \mathcal{L}^N\times\Delta$ the alternatives $x,y\in X$, x is said to be terminally stochastically dominated by y if there exists $K\in\{1,\ldots,m-1\}$ such that $\Pr(\{rk(y)\le K\}|(R_N,p)) > \Pr(\{rk(x)\le K\}|(R_N,p))$ and $\Pr(\{rk(y)\le r\}|(R_N,p)) \ge \Pr(\{rk(x)\le r\}|(R_N,p))$ for all $r\in\{K+1,\ldots,m\}$.

An ECC f on $\mathcal{R}$ is said to satisfy No-Terminal Stochastic Domination if for all $(R_N,p)\in\mathcal{R}$ and $x,y\in X$: [x is terminally stochastically dominated by y] implies $[x\notin f(R_N,p)]$.

Lemma 1: If an ECC f on a domain $\mathcal{R}$ satisfies IIS and No-Terminal Stochastic Domination, then for all $(R_N,p)\in\mathcal{R}$ it must be the case that $f(R_N,p)\subset Mm(R_N,p)$.

Proof: Suppose f on a domain $\mathcal{R}$ satisfies No-Terminal Stochastic Domination.

Let $(R_N,p)\in\mathcal{R}$ and $x\in f(R_N,p)$.

By IIS, we may without loss of generality suppose that support (p) = N.

If worstrk $(x, (R_N,p)) = 1$, then Mm $(R_N,p) = \{x\}$, and so $x\in Mm(R_N,p)$. Hence suppose $r = \text{worstrk}(x, (R_N,p)) > 1$.

Towards a contradiction, suppose that for $y\in Mm(R_N,p)$, it is the case that $\text{worstrk}(y, (R_N,p)) = \rho < r$.

Thus, $1 = \Pr(\{rk(y)\le\rho\}|(R_N,p)) > \Pr(\{rk(x)\le\rho\}|(R_N,p))$ and $1 = \Pr(\{rk(y)\le r\}|(R_N,p)) \ge \Pr(\{rk(x)\le r\}|(R_N,p))$ for all $r\in\{\rho+1,\ldots,m\}$.

But then y terminally stochastically dominates x, contradicting our assumption that f satisfies No-Terminal Stochastic Domination.

Thus, $x\in Mm(R_N,p)$, and hence $f(R_N,p)\subset Mm(R_N,p)$. Q.E.D.

4. The problem with Max-min and a possible refinement

The problem with $Mm(R_N,p)$ and any ECC that does not discriminate between states of nature which have positive probability is that they might over-emphasize the "extremely unlikely" to absurd extents, thereby denying the decision-maker the right to exercise one's discretion within reasonable limits.

The following example comes from Lahiri (2022).

Example 1: X = {x,y}, n = 2,

$$(R_N,p) = \begin{matrix} \frac{1}{100} & \frac{99}{100} \\ \end{matrix} \begin{bmatrix} x & y \\ y & x \end{bmatrix}$$

$Mm(R_N,p) = \{x,y\}$. But, does 'x' have any reason to be treated at par with 'y', when there is a 99% chance that 'y' is going to be preferred to 'x'?

Hence, we consider the procedure below.

The next notation will prove useful in what follows.

Given that $(R_N,p)\in \mathcal{L}^N\times\Delta$ and $x\in X$, the probability of the worst rank of x at (R_N,p) denoted $Pr(WS(x,R_N,p)) = \sum_{i\in WS(x,R_N,p)} p_i$.

An ECC on $\mathcal{R}$ is called the probabilistic max-min choice correspondence, denoted f^{PMm}, if for all $(R_N,p)\in \mathcal{R}$, $f^{PMm}(R_N,p) = \{x\in Mm(R_N,p)|\ Pr(WS(x,R_N,p)) \leq Pr(WS(y,R_N,p))$ for all $y\in Mm(R_N,p)\}$, i.e. $f^{PMm}(R_N,p)$ is the set of max-min alternatives with the least total probability of securing the worst rank at (R_N,p).

Thus, an ECC is f^{PMm} which at any (R_N,p) in the domain of the ECC chooses those max-min alternatives whose max-min rank occurs with the least probability, i.e. the chosen alternative are those max-min alternatives each of which occurs at its worst rank with the least probability. In other words, f^{PMm} minimizes "the probability" with which a max-min rank occurs.

Clearly, $f^{PMm}(R_N,p)$ for Example 1 is {y}.

In view of the fact that domain $\mathcal{R}$ is a subset of $\mathcal{L}^N\times\Delta$, given any $(R_N,p)\in \mathcal{R}$, it is not possible for two different alternatives to have the same worst state of nature at (R_N,p)".

Example 2: For m = 3 with X = {x,y,z} and n = 4, let (R_N,p) be defined as follows:

$$\begin{matrix} \frac{2}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{matrix}$$
$$\begin{bmatrix} x & y & z & z \\ y & x & y & x \\ z & z & x & y \end{bmatrix}$$

Here $f^{PMm}(R_N,p) = \{x,y\}$, but y is terminally stochastically dominated by x.

Thus, we are led to the following refinement of the Probabilistic Max-min, and hence a further refinement of Max-min, which for each extended preference profile (R_N,p) beginning with $f^{PMm}(R_N,p)$ iteratively chooses (from among the chosen alternatives from the previous stage) those alternatives which occur at any rank with the highest (cumulative) probability of securing a better rank, all the way up to the second rank.

Let k = worstrk (x, R_N,p) for $x\in Mm(R_N,p)$.

If k = 1, then $Mm(R_N,p) = f^{PMm}(R_N,p)$ is a singleton.

If $f^{PMm}(R_N,p)$ is a singleton then STOP.

If $f^{PMm}(R_N,p)$ is not a singleton then k > 1.

(This does not necessarily mean that if k > 1, then $f^{PMm}(R_N,p)$ is not a singleton).

Let $f^{PMm}(R_N,p)$ be denoted k- $f^{PMm}(R_N,p)$, where k > 1 and the cardinality of the set k- $f^{PMm}(R_N,p)$ is greater than one.

Thus, k- $f^{PMm}(R_N,p) = \{x\in Mm(R_N,p)|\ Pr(\{rk(x) = k\}|(R_N,p)) \leq Pr(\{rk(y) = k\}|(R_N,p))$ for all $y\in Mm(R_N,p)\} = \{x\in Mm(R_N,p)|\ Pr(\{rk(x) \leq k\text{-}1\}|(R_N,p)) \geq Pr(\{rk(y) \leq k\text{-}1\}|(R_N,p))$ for all $y\in Mm(R_N,p)\}$

If k = 2, then $f^{PMm}(R_N,p)$ = 2- $f^{PMm}(R_N,p)$ = {x∈Mm(R_N,p)| Pr({rk(x) = 2}|(R_N,p)) ≤ Pr({rk(y) = 2}|(R_N,p)) for all y∈$f^{PMm}(R_N,p)$} = {x∈Mm(R_N,p)| Pr({rk(x) = 1}|(R_N,p)) ≥ Pr({rk(y) = 1}|(R_N,p)) for all y∈$f^{PMm}(R_N,p)$}

Hence if k = 2, STOP.

Now suppose k > 2.

Having defined ρ- $f^{PMm}(R_N,p)$ with k ≥ ρ > 2 and the cardinality of the set ρ- $f^{PMm}(R_N,p)$ greater than one, let (ρ-1)- $f^{PMm}(R_N,p)$ = {x∈ρ- $f^{PMm}(R_N,p)$| Pr({rk(x) ≤ ρ - 2}|(R_N,p)) ≥ Pr({rk(y) ≤ ρ - 2}|(R_N,p)) for all y∈ρ- $f^{PMm}(R_N,p)$} = {x∈ρ- $f^{PMm}(R_N,p)$| Pr({rk(x) = ρ-1}|(R_N,p)) ≤ Pr({rk(y) = ρ-1}|(R_N,p)) for all y∈ρ- $f^{PMm}(R_N,p)$}.

Since this iterative process cannot go on indefinitely, we finally arrive at the set K^*- $f^{PMm}(R_N,p)$ with K^* ≥ 2 such that:

either (i) cardinality of K- $f^{PMm}(R_N,p)$ > 1 for all K∈{2,…,k}, in which case K^* = 2;

or (ii) K^* = max {K| K-$f^{PMm}(R_N,p)$ is a singleton}.

Let APMm (R_N,p) = K^*- $f^{PMm}(R_N,p)$ if x∈Mm (R_N,p) implies worstrk(x, (R_N,p)) > 1, and APMm (R_N,p) = Mm(R_N,p), otherwise.

The extended choice correspondence f^{APMm} on $\mathcal{R}$, defined as $f^{APMm}(R_N,p)$ = APMm (R_N,p) for all (R_N,p)∈$\mathcal{R}$, is referred to as the Ascending Probabilistic Max-min Extended Choice Correspondence on $\mathcal{R}$.

In example 3, APMm (R_N,p) = {x}.

5. An axiomatic characterization of the Ascending Probabilistic Max-min ECC

Lemma 2: (a) The ECC f^{APMm} on $\mathcal{R}$ satisfies No-Terminal Stochastic Domination. (b) Let f on $\mathcal{R}$ be an ECC that satisfies IIS and No-Terminal Stochastic Domination. Then f(R_N,p) ⊂ APMm(R_N,p) for all (R_N,p)∈$\mathcal{R}$.

Proof: Let (R_N,p)∈$\mathcal{R}$.

(a) By definition, $f^{APMm}(R_N,p)$ ⊂ $f^{PMm}(R_N,p)$ ⊂ Mm(R_N,p). Let x∈ $f^{APMm}(R_N,p)$ and suppose worstrk (x,(R_N,p)) = K. Towards a contradiction, suppose y terminally stochastically dominates x at (R_N,p).

If y∉Mm(R_N,p), then 1 = Pr({rk(x) ≤ K}|(R_N,p)) > Pr({rk(y) ≤ K}|(R_N,p)) and 1 = Pr({rk(x) ≤ k}|(R_N,p)) ≥ Pr({rk(y) ≤ k}|(R_N,p)) for all k∈{K+1,…,m}. This violates the requirement for y to terminally stochastically dominate x.

If y∈Mm(R_N,p), then 0 < Pr({rk(x) = K}|(R_N,p)) ≤ Pr({rk(y) = K}|(R_N,p)) and Pr({rk(x) ≤ k}|(R_N,p)) = Pr({rk(y) ≤ k}|(R_N,p)) = 1 for all k∈{K+1,…,m}, since x∈$f^{APMm}(R_N,p)$.

$0 < \Pr(\{rk(x) = K\}|(R_N,p)) \leq \Pr(\{rk(y) = K\}|(R_N,p))$ implies $\Pr(\{rk(x) \leq K\text{-}1\}|(R_N,p)) \geq \Pr(\{rk(y) \leq K\text{-}1\}|(R_N,p))$, since $\Pr(\{rk(x) \leq K\}|(R_N,p)) = 1 = \Pr(\{rk(y) \leq K\}|(R_N,p))$

Since y terminally stochastically dominates x, it must be the case that $\Pr(\{rk(y) \leq \rho\}|(R_N,p)) > \Pr(\{rk(x) \leq \rho\}|(R_N,p))$ for some $\rho < K\text{-}1$ and $\Pr(\{rk(y) \leq k\}|(R_N,p)) \geq \Pr(\{rk(x) \leq k\}|(R_N,p))$ for all $k > \rho$. This, in particular, implies $y \in f^{PMm}(R_N,p)$.

This contradicts our assumption that $x \in f^{APMm}(R_N,p)$.

Thus, f^{APMm} satisfies No-Terminal Stochastic Domination.

(b) Suppose f is an ECC on $\mathcal{R}$ that satisfies ISS and No-Terminal Stochastic Domination.

By lemma 1, for all $(R_N,p) \in \mathcal{R}$, it is the case that $f(R_N,p) \subset Mm(R_N,p)$.

Let $x \in f(R_N,p)$ and suppose worstrk $(x, (R_N,p)) = K$.

Since $x \in Mm(R_N,p)$, $\Pr(\{rk(x) \leq K\}|(R_N,p)) = 1$.

Towards a contradiction, suppose that there exists $y \in Mm(R_N,p)$ and $\rho \geq 1$ with $\rho \leq K\text{-}1$, satisfying $\Pr(\{rk(y) \leq \rho\}|(R_N,p)) > \Pr(\{rk(x) \leq \rho\}|(R_N,p))$ and $\Pr(\{rk(y) \leq k\}|(R_N,p)) \geq \Pr(\{rk(y) \leq k\}|(R_N,p))$ for all $k > \rho$.

This violates the requirement that f satisfies No-Terminal Stochastic Domination.

Thus, for $y \in Mm(R_N,p)$ and $\rho \geq 1$ with $\rho \leq K\text{-}1$, either (i) $\Pr(\{rk(y) \leq \rho\}|(R_N,p)) \leq \Pr(\{rk(x) \leq \rho\}|(R_N,p))$ in which case $x \in APMm(R_N,p)$, or (ii) $\Pr(\{rk(y) \leq \rho\}|(R_N,p)) > \Pr(\{rk(x) \leq \rho\}|(R_N,p))$, and for some $k > \rho$, in which case it must be that $\Pr(\{rk(y) \leq k\}|(R_N,p)) < \Pr(\{rk(y) \leq k\}|(R_N,p))$, the latter implying $y \notin APMm(R_N,p)$.

Since $APMm(R_N,p) \neq \phi$, (i) and (ii) imply that $x \in APMm(R_N,p)$.

Thus, $f(R_N,p) \subset APMm(R_N,p)$ as desired. Q.E.D.

Let $\mathcal{R}^{APMm} = \{(R_N,p) \in \mathcal{L}^N \times \Delta | APMm(R_N,p)$ is a singleton$\}$, i.e. the largest domain (of extended preference profiles) on which f^{PMm} is resolute.

An immediate and important consequence of Lemma 2 is the following Corollary.

Corollary to Lemma 2: A ECC f on $\mathcal{R}^{APMm}$ satisfies IIS and No-Terminal Stochastic Domination if and only if [f is resolute and $f = f^{APMm}$].

IIS and No-Terminal Stochastic Domination along with Probabilistic Neutrality can be used to establish an axiomatic characterization of the Ascending Probabilistic Max-min solution.

Proposition 1: An ECC f on $\mathcal{R}$ satisfies IIS, Probabilistic Neutrality and No-Terminal Stochastic Domination if and only if $f = f^{APMm}$.

Proof: It is easy to see that f^{APMm} on $\mathcal{R}$ satisfies Probabilistic Neutrality.

Hence, suppose that f on $\mathcal{R}$ satisfies IIS, Probabilistic Neutrality and No-Terminal Stochastic Domination.

Let $(R_N,p) \in \mathcal{R}$.

By Lemma 2, we know that $f(R_N,p) \subset APMm(R_N,p) = f^{APMm}(R_N,p)$.

If for $y \in Mm(R_N,p)$, the worst-rank $(y, (R_N,p)) = 1$, then $f^{APMm}(R_N,p)$ = Max-min Winners(M) is a singleton.

Thus, $f(R_N,p) = f^{APMm}(R_N,p)$.

Let $APMm(R_N,p) = K^*\text{-}f^{PMm}(R_N,p)$ with $K^* > 2$.

Then, $APMm(R_N,p) = K^*\text{-}f^{PMm}(R_N,p)$ is a singleton, so that $f(R_N,p) \subset APMm(R_N,p) = f^{APMm}(R_N,p)$ implies $f(R_N,p) = f^{APMm}(R_N,p)$.

Hence, suppose $K^* = 2$. Then for all $x,y \in APMm(R_N,p) = K^*\text{-}f^{PMm}(R_N,p)$, it is the case that $Pr(\{rk(y) = k\}|(R_N,p)) = Pr(\{rk(y) = k\}|(R_N,p))$ for all $k \in \{1,\ldots,m\}$.

By Probabilistic Neutrality, $x \in f(R_N,p)$ if and only if $y \in f(R_N,p)$.

Thus, $f(R_N,p) = f^{APMm}(R_N,p)$. Q.E.D.

Note: Given a strict ranking on X, the extended choice correspondence which for all $(R_N,p) \in \mathcal{R}$ chooses from $APMm(R_N,p)$, the alternative that is ranked best according to this strict ranking is singleton valued (resolute), satisfies IIS and No-Terminal Stochastic Domination, but does not satisfy Probabilistic Neutrality. The choice function that selects the whole set X for all $(R_N,p) \in \mathcal{R}$ satisfies IIS and Probabilistic Neutrality, but does not satisfy the No-Terminal Stochastic Domination.

6. Conclusion

The solution concept we suggest here violates the Condorcet consistency, which requires that if an alternative is preferred to all other alternatives in a pair-wise comparison, then such an alternative should be (the only alternative to be) chosen. This is easily seen for an extended preference profile with three alternatives: x, y, z, where x is ranked first with probability $\frac{501}{1000}$, and last with probability $\frac{499}{1000}$, whereas y is ranked second with probability 1. Our solution would select y, in spite of x being the Condorcet winner. The problem with x is its extreme volatility, and our solution concept protects the decision-maker from the not-unlikely adverse consequences that the choice of the Condorcet winner would expose him or her to.

An alternative way of proceeding with our analysis would be to use ordinal data matrices. An ordinal data matrix gives the probability with which each alternative is assigned each rank. Clearly, such a matrix is a bi-stochastic matrix of rational numbers, assuming that the probability with which each state of nature occurs is a rational number. The Birkhoff-von Neumann theorem states that every bi-stochastic matrix is an expected ranking matrix, though there may be more than one probability distribution over strict rankings that lead to the same expected ranking matrix. Many choice procedures based on preference profiles, including the procedure we discuss here, can be stated in terms of data contained in such matrices.

Aleskerov and Subochev (2013) study the representation of binary relations on finite sets by logical matrices. They are largely concerned with the “preferred with probability at least half” relation.

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